1. Determine whether each statement is true or false. If the statement is true, cite your reasoning. If it is false, provide an example showing the statement to be false.

(a) If $f'(x) < 0$ on the interval $(0, 4)$ then $f(x)$ is concave down on the interval $(0, 4)$.

Solution: False. The following function has $f'(x) < 0$ on the interval $(0, 4)$ but is not concave down on the interval $(0, 4)$. 

![Graph of f(x) with f'(x) < 0 on (0, 4) but not concave down]

(b) If $f(x)$ is continuous at $x = 0$ then $f(x)$ is differentiable at $x = 0$.

Solution: False. The function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$ since its graph has a corner at $x = 0$. 


2. Use the definition of the derivative (i.e. the limit definition) to compute: \( \frac{d}{dx}(\sqrt{x+1}) \).

Solution:

\[
\frac{d}{dx}(\sqrt{x+1}) = \lim_{h \to 0} \left( \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{(\sqrt{x+h+1} - \sqrt{x+1})(\sqrt{x+h+1} + \sqrt{x+1})}{h(\sqrt{x+h+1} + \sqrt{x+1})} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} \right)
\]
\[
= \lim_{h \to 0} \left( \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} \right)
\]
\[
= \frac{1}{1 + \frac{1}{2}\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}
\]
3. The graphs of $f$, $f'$, and $f''$ are below. Identify each curve and explain your choices.

- $f''(x)$ - has zeros where $f'$ has horizontal tangents.
- $f(x)$
- $f'(x)$ - has zeros where $f$ has horizontal tangents.
4. Let \( f(x) = 3x^5 - 5x^3 + 3 \)

(a) Compute \( f'(x) \) and \( f''(x) \). You do not have to use the limit definition of the derivative here.

*Solution:* 
\[
f'(x) = 15x^4 - 15x^2 \\
f''(x) = 60x^3 - 30x
\]

(b) List all the critical points of \( f(x) \).

*Solution:* The critical points are the solutions to \( f'(x) = 0 \).
\[
15x^4 - 15x^2 = 0 \\
15x^2(x^2 - 1) = 0 \\
x^2(x^2 - 1) = 0
\]

So the critical points are \( x = 0, -1, 1 \).

(c) Find the intervals on which \( f(x) \) is increasing and the intervals on which \( f(x) \) is decreasing.

*Solution:* We need to check the sign of \( f'(x) \) on the following intervals: \((-\infty, -1)\), \((-1, 0)\), \((0, 1)\), and \((1, \infty)\). Notice:
\[
f'(-2) > 0 \\
f'(-1/2) < 0 \\
f'(1/2) < 0 \\
f'(2) > 0
\]

This tells us that the function \( f \) is increasing on \((-\infty, -1)\) and \((1, \infty)\). The function \( f \) is decreasing on \((-1, 0)\) and \((0, 1)\).
(d) Find the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.

Solution: Notice:

\[ f''(x) = 0 \]
\[ 60x^3 - 30x = 0 \]
\[ 30x(2x^2 - 1) = 0 \]

We need to check the sign of $f''(x)$ on the intervals $(-\infty, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, 0)$, $(0, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, \infty)$.

On $(-\infty, -\frac{1}{\sqrt{2}})$, $f''(x) < 0$.
On $(-\frac{1}{\sqrt{2}}, 0)$, $f''(x) > 0$.
On $(0, \frac{1}{\sqrt{2}})$, $f''(x) < 0$.
On $(\frac{1}{\sqrt{2}}, \infty)$, $f''(x) > 0$.

Thus the function $f$ is concave down on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(0, \frac{1}{\sqrt{2}})$. It is concave up on $(-\frac{1}{\sqrt{2}}, 0)$ and $(\frac{1}{\sqrt{2}}, \infty)$.

(e) From your list of critical points, determine which are actually local maxima and which are actually local minima (for each, be sure to justify why it is a max/min or neither).

Solution: From part b, we know the critical points are -1, 0, and 1. Since $f''(-1) = 30(-1)(2(1) - 1) < 0$, the second derivative test tells us that -1 is a local maximum. Similarly, $f''(1) = 30(1)(2 - 1) > 0$ so the second derivative test tells us that 1 is a local minimum.

From part c, we know that $f'(x) < 0$ to the immediate left of 0 and $f'(x) < 0$ to the immediate right of 0. The first derivative test tells us that 0 is neither a local minimum nor a local maximum.
5. Sketch the graph of a single function $g(x)$ with the following properties:

- $g'(1) = 0,$
- $g'(0)$ does not exist,
- $g'(x) < 0$ when $x < -1$ and when $x > 1$,
- $g'(x) > 0$ when $0 < x < 1$,
- $g''(x) > 0$ when $x < -1$,
- $g''(x) < 0$ when $\frac{1}{2} < x < 2$. 
6. Find the equation of the tangent line to \( x^2 + xy + y^2 = e^{y-2} + 6 \) at \((1,2)\).

Solution:

\[
2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = e^{y-2} \frac{dy}{dx}
\]

\[
2x + y = \frac{dy}{dx}(-x - 2y + e^{y-2})
\]

\[
\frac{dy}{dx} = \frac{2x - y}{-x - 2y + e^{y-2}}
\]

The slope of the tangent line at \((1,2)\) is \( \frac{2(1) + 2}{-1 - 2(2) + e^{2-2}} = -1 \).

Then the tangent line at \((1,2)\) is \( y - 2 = -(x - 1) \).

7. Suppose \( h(x) = (f(x))^2 g(x) \), where \( f(x) \) and \( g(x) \) are differentiable functions. Find \( h'(1) \) when \( f(1) = 1, \ g(1) = -1, \ f'(1) = 2, \) and \( g'(1) = 3 \).

Solution:

\[
h'(x) = \frac{d}{dx}([f(x)]^2 g(x) + [f(x)]^2 g'(x))
\]

\[
h'(x) = 2f(x)f'(x)g(x) + [f(x)]^2 g'(x)
\]

\[
h'(1) = 2f(1)f'(1)g(1) + [f(1)]^2 g'(1)
\]

\[
h'(1) = 2(1)(2)(-1) + [1]^2(3)
\]

\[
h'(1) = -1
\]
8. Compute the following derivatives:

(a) Let \( f(x) = (\sin(x))^{76} \). Find \( f'(x) \).

\[ Solution: \]
\[ 76(\sin(x))^{75}(\cos(x)) \]

(b) Let \( f(x) = 10^x \ln(x) \). Find \( f'(x) \).

\[ Solution: \]
\[ \ln(10)10^x \ln(x) + \frac{10^x}{x} \]

(c) Differentiate \( \left( \frac{e^{3x}}{5 \cos(x)} \right) \).

\[ Solution: \]
\[ \frac{5 \cos(x)(3e^{3x}) - e^{3x}(-5 \sin(x))}{[5 \cos(x)]^2} \]
(d) Find \( \frac{d}{dx}(\sqrt[5]{x^2}(\tan^2(x) + e^4)) \).

\[ \text{Solution:} \]
\[ \frac{2}{5}x^{-3/5}(\tan^2(x) + e^4) + x^{2/3}(2\tan(x)\sec^2(x)) \]

(e) Find \( \frac{d}{dx}(\sin(x^4 + 50x^2 + 2)) \).

\[ \text{Solution:} \]
\[ \cos(x^4 + 50x^2 + 2)(4x^3 + 100x) \]