1. Give the MATHEMATICAL definitions of

(a) The limit of the function $f(x)$, as $x$ approaches $a$ from the right, equals $L$.

Solution: We can make the values $f(x)$ arbitrarily close to $L$ by taking $x$ suffi-
iciently close to $a$ and $x > a$. □

(b) $f(x)$ is continuous at $x = a$.

Solution:

$$\lim_{x \to a} f(x) = f(a).$$

□

(c) The derivative of $f(x)$ at the point $x = a$.

Solution: Either definition is acceptable:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$ □

2. (a) Sketch the graph of the function

$$g(x) = \begin{cases} 
1 - x^2 & : x < 0 \\
1 & : x = 0 \\
2x - 2 & : 0 < x \leq 1 \\
\sqrt{x - 1} & : x > 1.
\end{cases}$$
(b) What is \( \lim_{x \to 1} g(x) \)?

**Solution:**

\[
\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (2x - 2) \\
= 2(1) - 2 \quad (b/c \ 2x - 2 \text{ is continuous at } x = 1.) \\
= 0.
\]

\[
\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \sqrt{x - 1} \\
= \sqrt{1 - 1} \quad (b/c \ \sqrt{x - 1} \text{ is continuous at } x = 1.) \\
= 0.
\]

Therefore, since \( \lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) = 0 \), \( \lim_{x \to 1} g(x) = 0 \).

(c) What is \( \lim_{x \to 0^+} g(x) \)?

**Solution:**

\[
\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (2x - 2) \quad (b/c \ g(x) = 2x - 2 \text{ to the immediate right of } 0.) \\
= 2(0) - 2 \quad (b/c \ 2x - 2 \text{ is continuous at } x = 1.) \\
= -2.
\]

(d) What is \( \lim_{x \to 0^-} g(x) \)?

**Solution:**

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (1 - x^2) \quad (b/c \ g(x) = 1 - x^2 \text{ to the immediate left of } 0.) \\
= 1 - 0^2 \quad (b/c \ 1 - x^2 \text{ is continuous at } x = 1.) \\
= 1.
\]

(e) Is \( g(x) \) continuous at \( x = 0 \)?
3. Determine whether each statement is true or false for arbitrary functions \( f(x) \) and \( g(x) \). If the statement is true, cite your reasoning. If it is false, provide an example showing the statement to be false.

(a) If \( \lim_{x \to 0} g(x) = 0 \), then \( \lim_{x \to 0} \frac{f(x)}{g(x)} \) does not exist.

**Solution:** False. An example showing that this is false is \( f(x) = x^2 \) and \( g(x) = x \):

\[
\lim_{x \to 0} g(x) = 0
\]

but

\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0
\]

which clearly exists.

(b) If \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) exist, then \( \lim_{x \to 0} (f(x)g(x)) \) exists.

**Solution:** True. The product limit law. More specifically, if \( \lim_{x \to 0} f(x) = L \) and \( \lim_{x \to 0} g(x) = M \), then \( \lim_{x \to 0} (f(x)g(x)) = LM \).

4. Compute the following limits; justify your answers. You are allowed to use any rules we’ve shown in class; quote the rules you use. If a limit does not exist, explain why.

(a) \( \lim_{x \to 1} \frac{x^2 + 2x + 1}{\sqrt{x + 1}} \)

**Solution:** \( \frac{x^2 + 2x + 1}{\sqrt{x + 1}} \) is continuous at \( x = 1 \). Then

\[
\lim_{x \to 1} \frac{x^2 + 2x + 1}{\sqrt{x + 1}} = \frac{(1)^2 + 2(1) + 1}{\sqrt{(1) + 1}} = \frac{4}{\sqrt{2}}
\]

(b) \( \lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 3x - 4} \)
Solution:

\[
\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{(x - 4)(x - 1)}{(x - 4)(x + 1)} \\
= \lim_{x \to 4} \frac{x - 1}{x + 1} \quad \text{(b/c the limit doesn’t see } x = 4) \\
= \frac{4 - 1}{4 + 1} \quad \text{(b/c } \frac{x - 1}{x + 1} \text{ is continuous at } x = 4.) \\
= \frac{3}{5}
\]

(c) \(\lim_{x \to 0} \frac{|x|}{x}\)

Solution: Since \(|x|\) is a piecewise function, we will compute the left-hand limit and the right-hand limit to determine the limit of this function at \(x = 0\).

\[
\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -\frac{x}{x} = \lim_{x \to 0^-} -1 = -1.
\]

\[
\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1.
\]

Since \(\lim_{x \to 0^+} \frac{|x|}{x} \neq \lim_{x \to 0^-} \frac{|x|}{x}\), \(\lim_{x \to 0^+} \frac{|x|}{x}\) does not exist.

(d) \(\lim_{x \to 3^+} [\cos(x^2) + (\cos(x))^2]\)

Solution: \([-\cos(x^2) + (\cos(x))^2]\) is continuous at \(x = 3\). Then the substitution rule for limits of continuous functions gives us:

\[
\lim_{x \to 3^+} [\cos(x^2) + (\cos(x))^2] = [\cos(3^2) + (\cos(3))^2] \\
= \cos(9) + \cos^2(3)
\]

(e) \(\lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h}\)
Solution:

\[
\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \to 0} \left( \frac{\sqrt{1+h} - 1}{h} \right) \left( \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{1 + h - 1}{h(\sqrt{1 + h} + 1)} \right)
\]

\[
= \lim_{h \to 0} \left( \frac{h}{h(\sqrt{1 + h} + 1)} \right) \quad (b/c \text{ limit doesn’t see } x = 1.)
\]

\[
= \lim_{h \to 0} \left( \frac{1}{\sqrt{1 + h} + 1} \right)
\]

\[
= \frac{1}{2} \quad (b/c \text{ continuous at } h = 0.)
\]

5. The distance traveled by a particle at time \( t \) is given (in meters) by \( s(t) = t^2 + 1 \).

(a) Find the average velocity of the particle from \( t = 1 \) to \( t = 4 \). Graph the position function. Draw the secant line whose slope is the requested average velocity.

Solution: The average velocity of the particle from \( t = 1 \) to \( t = 4 \) is

\[
\frac{s(4) - s(1)}{4 - 1} = \frac{17 - 2}{2} = \frac{15}{2}
\]

(b) Find the average velocity of the particle from \( t = 1 \) to \( t = t_0 \) (where \( t_0 \) is some number greater than 0).

Solution: The average velocity of the particle from \( t = 1 \) to \( t = t_0 \) is

\[
\frac{s(t_0) - s(1)}{t_0 - 2} = \frac{(t_0^2 + 1) - 2}{t_0 - 1} = \frac{t_0^2 - 1}{t_0 - 1}
\]

(c) The answer to part (b) is a function of \( t_0 \). Find the limit of that function as \( t_0 \to 1 \).

Solution:

\[
\lim_{t_0 \to 1} \frac{t_0^2 - 1}{t_0 - 1} = \lim_{t_0 \to 1} \frac{(t_0 - 1)(t_0 + 1)}{t_0 - 1}
\]

\[
= \lim_{t_0 \to 1} (t_0 + 1) \quad (b/c \text{ the limit doesn’t see } t_0 = 1.)
\]

\[
= 2
\]
(d) Using your answer in part (c), find the equation of the tangent line to \( y = t^2 + 1 \) at \( t = 1 \).

\[
\text{Solution: When } t = 1, \quad y = 1^2 + 1 = 2. \text{ The equation of the tangent line is } \frac{y - 2}{t - 1} = 2(t - 1) \text{ since the answer to part (c) is the slope of the tangent line at } t = 1.\]

6. Prove that
\[ f(x) = \cos(\sqrt{x} + 2) \]

is continuous for \( x > 0 \).

\[
\text{Solution: -} \quad \cos x \text{ is a continuous function for all values of } x.
\]
\[
- \sqrt{x} \text{ is continuous for all values of } x \geq 0 \text{ since it is a root function.}
\]
\[
- \text{ The sum } \sqrt{x} + 2 \text{ is continuous since it is the sum of two continuous functions.}
\]
\[
- \text{ The composition of continuous functions is continuous, thus } \cos(\sqrt{x}+2) \text{ is continuous.}
\]

7. Let \( f(x) = \frac{2x}{x+3} \). Find \( f'(2) \) using the limit definition of the derivative. Show the steps of your computation.

\[
\text{Solution:}
\]
\[
f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{2(2+h)}{(2+h)+3} - \frac{2(2)}{2+3}}{h} = \lim_{h \to 0} \frac{\frac{4+2h}{5+h} - \frac{4}{5}}{h} = \lim_{h \to 0} \frac{(4+2h)(5) - 4(5+h)}{5(5+h)} = \lim_{h \to 0} \frac{20+10h-20-4h}{5(5+h)} \]
\[
= \lim_{h \to 0} \frac{6h}{5(5+h)} (b/c \text{ limit doesn’t see } h = 0.)
\]
\[
= \lim_{h \to 0} \frac{6}{5(5+h)} (b/c \frac{6}{5(5+h)} \text{ is continuous at } h = 0.) \]
\[
= \frac{6}{25}
\]
8. Sketch the graph of a function for which \( f(0) = 0 \), \( f'(0) = 3 \), \( f'(1) = 0 \), and \( f'(2) = -1 \).

9. Show that there is a root of the function \( p(t) = 10 + t - t^7 \) on the interval \([0, 10]\).

   \textit{Solution:} Since \( p(t) \) is a polynomial function it is continuous on the interval \([0, 10]\).

   \[ p(0) = 10 + 0 - 0^7 = 10 \]
   \[ p(10) = 10 + 10 - 10^7 \] which is a very large negative number.

   Since \( p(10) < 0 < p(0) \) and since \( p(t) \) is continuous on this interval, the Intermediate Value Theorem tells us that there exists a number \( c \) in the interval \((0, 10)\) that has \( p(c) = 0 \). In other words, \( c \) is a root for this function.