1. (12 points) Give the MATHEMATICAL definitions of

(a) The function $f(x)$ is continuous on the interval $[a,b]$.

Solution: $f'(c) = \lim_{x \to c} f(x)$ for all $c$ in $[a,b]$. OR: $f$ is continuous at every point $x = c$ in the interval $[a,b]$. \[ \square \]

(b) The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$.

Solution: The line through $P$ with slope

$$ m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} $$

provided that this limit exists. \[ \square \]

(c) The limit of the function $f(x)$, as $x$ goes to $a$, equals $L$.

Solution: We can make the values of $f(x)$ arbitrarily close to $L$ by taking $x$ to be sufficiently close to $a$, $x \neq a$. \[ \square \]

2. Show all your reasoning in the problems below. Let

$$ g(x) = \begin{cases} 
  x & : x \leq -\pi \\
  \sin(x) & : -\pi < x < \pi \\
  1 & : x = \pi \\
  2x - 2\pi & : x > \pi. 
\end{cases} $$

(a) (4 points) Graph $g(x)$. Please draw the graph as accurately as possible and with labels.
(b) (3 points) Find $\lim_{x \to (-\pi)^-} g(x)$.

Solution:

$$\lim_{x \to (-\pi)^-} g(x) = \lim_{x \to (-\pi)^-} x \quad (\text{b/c } g(x) = x \text{ to the immediate left of } -\pi)$$

$$= \boxed{-\pi}$$

\(\square\)

(c) (3 points) Find $\lim_{x \to (-\pi)^+} g(x)$.

Solution:

$$\lim_{x \to (-\pi)^+} g(x) = \lim_{x \to (-\pi)^+} \sin(x) \quad (\text{b/c } g(x) = \sin(x) \text{ to the immediate right of } -\pi)$$

$$= \sin(-\pi) \quad (\text{b/c } \sin(x) \text{ is a continuous function})$$

$$= \boxed{0}$$

\(\square\)

(d) (4 points) Find $\lim_{x \to \pi} g(x)$.

Solution:

$$\lim_{x \to \pi^-} g(x) = \lim_{x \to \pi^-} \sin(x) \quad (\text{b/c } g(x) = \sin(x) \text{ to the immediate left of } \pi)$$

$$= \sin(\pi) \quad (\text{b/c } \sin(x) \text{ is a continuous function})$$

$$= 0$$

$$\lim_{x \to \pi^+} g(x) = \lim_{x \to \pi^+} (2x - 2\pi) \quad (\text{b/c } g(x) = 2x - 2\pi \text{ to the immediate right of } \pi)$$

$$= 2\pi - 2\pi \quad (\text{b/c } 2x - 2\pi \text{ is a continuous function})$$

$$= 0$$
Thus \( \lim_{x \to \pi} g(x) = 0 \)

(e) (2 points) Is \( g(x) \) continuous at \( x = \pi \)?

**Solution:** No. Because \( 1 = g(\pi) \neq \lim_{x \to \pi} g(x) = 0. \)

3. (6 points) Determine whether each statement is true or false for arbitrary functions \( f(x) \) and \( g(x) \). If the statement is true, cite your reasoning. If it is false, provide an example showing the statement to be false.

(a) If \( f(0) \) is undefined, then \( \lim_{x \to 0} f(x) \) does not exist.

**Solution:** False. The simplest example is \( f(x) = x / x \)

which is 1 everywhere but at \( x = 0 \) where it is undefined.

(b) If \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) do not exist, then \( \lim_{x \to 0} (f(x) - g(x)) \) does not exist.

**Solution:** False. Let \( f(x) = g(x) = 1 / x \). Then \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) do not exist but \( \lim_{x \to 0} (f(x) - g(x)) = \lim_{x \to 0} 0 = 0 \) which clearly exists.

4. Compute the following limits; justify your answers. You are allowed to use any rules we’ve shown in class; quote the rules you use. If a limit does not exist, explain why.

(a) (5 points) \( \lim_{x \to (-1)} \frac{x^4 - 3x^2 - 8}{x^2 - \sqrt{4x} + 8} \)

**Solution:**

\[
\lim_{x \to (-1)} \frac{x^4 - 3x^2 - 8}{x^2 - \sqrt{4x} + 8} = \frac{(-1)^4 - 3(-1)^2 - 8}{(-1)^2 - \sqrt{4(-1)} + 8} \quad (b/c \text{ this function is continuous at } x = -1)
\]

= \[
\frac{1 - 3 - 8}{1 - 8}
\]

= \[
\frac{-10}{-7}
\]

= \boxed{10}

(b) (5 points) \( \lim_{x \to (-2)} \frac{x^2 - 3x - 10}{x^2 + x - 2} \)
Solution:

\[
\lim_{x \to -2} \frac{x^2 - 3x - 10}{x^2 + x - 2} = \lim_{x \to -2} \frac{(x - 5)(x + 2)}{(x - 1)(x + 2)}
\]
\[
= \lim_{x \to -2} \frac{x - 5}{x - 1} \quad (\text{b/c the limit doesn’t see } x = -2)
\]
\[
= \frac{(-2) - 5}{(-2) - 1} \quad (\text{b/c } \frac{x - 5}{x - 1} \text{ is continuous at } x = -2)
\]
\[
= \frac{7}{3}
\]

(c) \(5\) points \(\lim_{x \to 0}(\tan(x^2) + e^2)^2\)

Solution:

\[
\lim_{x \to 0}(\tan(x^2) + e^2)^2 = (\tan((0)^2) + e^2)^2 \quad (\text{b/c this function is continuous at } x = 0)
\]
\[
= (0 + 1)^2
\]
\[
= 1
\]

(d) \(5\) points \(\lim_{h \to 0} \frac{h + \sqrt{4 + h} - 2}{h}\)

Solution:

\[
\lim_{h \to 0} \frac{h + \sqrt{4 + h} - 2}{h} = \lim_{h \to 0} \left(\frac{\sqrt{4 + h} - (2 - h)}{h}\right) \left(\frac{\sqrt{4 + h} + (2 - h)}{\sqrt{4 + h} + (2 - h)}\right)
\]
\[
= \lim_{h \to 0} \frac{4 + h - (2 - h)^2}{h(\sqrt{4 + h} + (2 - h))}
\]
\[
= \lim_{h \to 0} \frac{h(5 - h)}{h(\sqrt{4 + h} + (2 - h))}
\]
\[
= \lim_{h \to 0} \frac{5 - h}{\sqrt{4 + h} + (2 - h)} \quad (\text{b/c limit doesn’t see } h = 0)
\]
\[
= \frac{5}{\sqrt{4} + 2} \quad (\text{b/c this function is continuous at } h = 0)
\]
\[
= \frac{5}{4}
\]

5. Let

\(f(x) = 3x^2 + 6.\)
(a) (10 points) Using the limit definition of the derivative at a point, compute \( f'(a) \).

**Solution:**

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

\[
= \lim_{h \to 0} \frac{(3(a + h)^2 + 6) - (3a^2 + 6)}{h}
\]

\[
= \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 + 6 - 3a^2 - 6}{h}
\]

\[
= \lim_{h \to 0} \frac{6ah + 3h^2}{h}
\]

\[
= \lim_{h \to 0} 6a + 3h \quad \text{ (b/c limit doesn’t see h=0)}
\]

\[
= 6a \quad \text{ (b/c 6a+3h is continuous at h=0)}.
\]

(b) (5 points) Find the equation of the tangent line to the curve at \( x = 1 \).

**Solution:** The slope of the tangent line at \( x = 1 \) is \( f'(1) = 6(1) \) from above. The point that the line passes through is \((1, f(1)) = (1, 9)\). Thus the equation of the tangent line to the curve at \( x = 1 \) is \( y - 9 = 6(x - 1) \).

(c) (5 points) Suppose that \( f(x) \) represents the position of a particle at time \( x \).

i. Explain, in words, what the slope of the secant line between the points (0,6) and (2,18) means.

**Solution:** The slope of the secant line between these points is the average velocity between the times \( x = 0 \) and \( x = 2 \).

ii. Explain, in words, what the slope of the tangent line to the curve at the point (2,18) means.

**Solution:** The slope of the tangent line to the curve at the point (2,18) is the instantaneous velocity time \( x = 2 \).
6. \((10 \text{ points})\) Let \(f(x) = \frac{x^2}{x+1}\). Find \(f'(1)\) using the limit definition of the derivative. Show the steps of your computation.

\[
\begin{align*}
\text{Solution:} \\
& f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\
& = \lim_{h \to 0} \frac{(1+h)^2}{(1+h)+1} - \frac{1^2}{1+1} \\
& = \lim_{h \to 0} \frac{(1+h)^2}{2+h} - \frac{1}{2} \\
& = \lim_{h \to 0} \frac{2(1+h)^2 - (2+h)}{2(2+h)} \\
& = \lim_{h \to 0} \frac{2 + 4h + 2h^2 - 2 - h}{2(2+h)} \\
& = \lim_{h \to 0} \frac{3h + 2h^2}{2h(2+h)} \\
& = \lim_{h \to 0} \frac{3 + 2h}{2(2+h)} \quad (\text{b/c limit doesn’t see } h=0) \\
& = \frac{3 + 2(0)}{2(2 + 0)} \quad (\text{b/c this function is continuous at } h=0) \\
& = 3/4
\end{align*}
\]

7. \((5 \text{ points})\) Prove that

\[
f(x) = \begin{cases} 
\cos(x) & : \ x \leq 0 \\
ed^{\sqrt{x}} & : \ x > 0 
\end{cases}
\]

is continuous on \((-\infty, \infty)\). Show all your reasoning.

\text{Solution:} \text{ When } x < 0, \ f(x) = \cos(x) \text{ which is continuous since it is a trig function.}\n
\text{When } x > 0, \sqrt{x} \text{ is a continuous function since it is a root function. } e^x \text{ is also continuous here because it is an exponential function. Further, } e^{\sqrt{x}} \text{ is continuous since it is the combination of continuous functions.}\n
When $x = 0$, we must check that $f(0) = \lim_{x \to 0} f(x)$. Note: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x) = 1$. Notice also that $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\sqrt{x}} = e^0 = 1$. Thus $\lim_{x \to 0} f(x) = 1 = f(0)$. Thus $f$ is continuous at $x = 0$.

8. (6 points) Jennifer’s position at time $t$ relative to Stone Cold Creamery is given by the function
\[ p(t) = 5 + 2t - t^2 \]
(positive indicates west and negative east). Explain why Jennifer must have visited the creamery between the hours of $t = 0$ and $t = 4$. Justify your answer with results from class!

Solution: The function $p(t)$ is continuous on the interval $[0,4]$ because it is a polynomial. Notice that $p(0) = 5 + 0 - 0 = 5$ and $p(4) = 5 + 8 - 16 = -3$. Since $p(4) = -3 < 0 < 5 = p(0)$, the Intermediate Value Theorem tells us that there is some number $c$ so that $p(c) = 0$. Thus at time $c$, Jennifer must have been at the Creamery.

9. (5 points) Sketch the graph of a continuous function that has the following properties:
- $f(0) = 1$
- $f(1) = 0$
- $f'(-1) = -2$
- $f'(0) = 1$
- $f'(1) = 0$.

Please put labels on your axes.

There are many possible solutions to this problem. See me if you’d like to see one of them.