Lecture Outline (Limits at Infinity)  
Monday, March 10

Announcement

- Office hours: March 11 (Tuesday) 7pm-10pm, and March 16 (Sunday) 7pm-10pm. They are tentatively scheduled in 380-T. In case they will be moved to a different room, a note will be posted on the door of 380-T.
- In-class review session on Friday.
- Homework will still be assigned this week. However, they will neither be collected or graded. They are solely for your preparation for the final exam.

Recap

Last time, we talked about the correspondence between infinite limits and vertical asymptotes. Remember vertical asymptotes only occur where the function is discontinuous. A function can have several vertical asymptotes.

Motivations

Example 1. Consider the function $f(x) = \frac{1}{x}$.

We can observe the following from its graph:

1. As $x$ is getting larger and larger positively, the values of $\frac{1}{x}$ gets closer and closer to 0. Symbolically, we write $\lim_{x \to \infty} \frac{1}{x} = 0$.

2. Similarly, as $x$ is getting larger and larger negatively, the values of $\frac{1}{x}$ gets closer and closer to 0. Symbolically, we write $\lim_{x \to -\infty} \frac{1}{x} = 0$.

3. We say the line $y = 0$ (i.e. $x$-axis) is a horizontal asymptote, which indicates that the line is approached by the graph of the function when the magnitude of $x$ is getting large.

Example 2. Consider the function $f(x) = e^x$. 
We can observe the following from its graph:

1. As $x$ is getting larger and larger negatively, the values of $e^x$ gets closer and closer to 0. So we say $\lim_{x \to -\infty} e^x = 0$.

2. As $x$ is getting larger and larger positively, the function $e^x$ grows unboundedly in the positive direction. We say $\lim_{x \to \infty} e^x = \infty$.

3. The limit of $e^x$ when $x \to -\infty$ tells us that $y = 0$ is a horizontal asymptote.

**Example 3.** Consider the function $f(x) = x^3$.

We can observe the following from its graph:

1. $\lim_{x \to \infty} x^3 = \infty$.

2. $\lim_{x \to -\infty} x^3 = -\infty$.

3. Since the function doesn’t approach any specific number when the magnitude of $x$ is large, the function doesn’t have any horizontal asymptote.

**Example 4.** Consider the function $f(x) = \sin x$.

We can observe the following from its graph:

1. As $x$ gets larger and larger positively, the function is always oscillating between $-1$ and 1. It neither approaches a certain number, nor does it grow unboundedly. In this case, we can only say $\lim_{x \to \infty} \sin x$ does not exist.

2. Similarly, in the other direction, we say $\lim_{x \to -\infty} \sin x$ does not exist.

3. Obviously, there’s no horizontal asymptote.

**Example 5.** Consider the function $f(x) = \arctan x$.

We can observe the following from its graph:

1. $\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$.

2. $\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$.

3. This is an example of a function which has two horizontal asymptotes. Since both $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are approached by the graph, both of them are horizontal asymptotes.
Summary

The behavior of a function $f(x)$ as $x$ approaches $\infty$ has three different cases:

1. It approaches a certain finite number $L$. We say $\lim_{x \to \infty} f(x) = L$, and $y = L$ is a horizontal asymptote.

2. It grows unboundedly in either the positive or the negative direction. We say $\lim_{x \to \infty} f(x) = \infty$ or $-\infty$.

3. It doesn’t belong to either of the above cases. We can say no more than $\lim_{x \to \infty} f(x)$ does not exist.

Similarly, we can talk about the behavior of a function as $x$ approaches $-\infty$. The conclusion is completely parallel to above.

Computations

In order to do algebraic computations without looking at graphs, we have the following 4 principles:

1. We can always make use of the known example $\lim_{x \to \pm \infty} \frac{1}{x} = 0$.

2. Limit laws are still valid: taking limits commutes with addition, subtraction, multiplication, division, taking power and taking root.

3. Usual techniques in limit computations are still applicable. For example, rationalizing numerators.

4. One more supplementary technique: divide both the numerator and denominator of a fraction by the highest power of $x$ appearing in the denominator. See examples below.

Example 6. Compute $\lim_{x \to \infty} \frac{x^2 + 3}{x^2 - x}$.

Solution. We need to divide both the numerator and the denominator by the highest power of $x$ appearing in the denominator, which is $x^2$.

$$\lim_{x \to \infty} \frac{x^2 + 3}{x^2 - x} = \lim_{x \to \infty} \frac{\left(\frac{x^2 + 3}{x^2}\right)}{\left(\frac{x^2}{x^2}\right) - \left(\frac{x}{x^2}\right)} = \lim_{x \to \infty} \frac{1 + \frac{3}{x^2}}{1 - \frac{1}{x}}$$

$$= \left(\lim_{x \to \infty} \frac{1 + \frac{3}{x^2}}{1 - \frac{1}{x}}\right) = \frac{1 + 3(\lim_{x \to \infty} \frac{1}{x})^2}{1 - (\lim_{x \to \infty} \frac{1}{x})} = \frac{1 + 3 \cdot 0^2}{1 - 0} = 1$$
Example 7. Find all horizontal asymptotes of the function \( f(x) = \frac{x^2 + 3}{x^2 - x} \).

Solution. In order to find horizontal asymptotes, we only need to compute the limits of the function as \( x \to \infty \) or \( x \to -\infty \). If the results are finite numbers, the function has horizontal asymptotes.

We have computed in the previous example that \( \lim_{x \to \infty} \frac{x^2 + 3}{x^2 - x} = 1 \). Using the same method, you can find \( \lim_{x \to -\infty} \frac{x^2 + 3}{x^2 - x} = 1 \) (do the computation on your own!). Therefore, \( y = 1 \) is the only horizontal asymptotes of the function.

Example 8. Compute \( \lim_{x \to \infty} (\sqrt{x^2 + 1} - x) \).

Solution. We need to rationalize the numerator first. The denominator in this problem is implicitly given by 1. After that we need to divide both the numerator and the denominator by the highest power of \( x \) in the denominator.

\[
\lim_{x \to \infty} (\sqrt{x^2 + 1} - x) = \lim_{x \to \infty} \frac{\sqrt{x^2 + 1} - x}{1} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{1}{\frac{x}{\sqrt{x^2 + 1} + x}} = \lim_{x \to \infty} \frac{1}{\frac{1}{x} \left( \sqrt{1 + \frac{1}{x^2}} + 1 \right)} = 0
\]

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Upcoming

On Wednesday, we are gonna do our last new topic which is l’Hospital’s rule. There is an interesting story about this rule which says this rule is actually discovered by Bernoulli. Why is it under the name of l’Hospital? Read the story on page 305 of textbook.