Announcement

Quiz 8 on Monday.

Upshot

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Motivation

Example 1. Let \( f(x) = \frac{1}{x^2} \). Compute \( \lim_{x \to a} \frac{1}{x^2} \).

You should be able to observe the following from the graph:

- \( f(x) \) is continuous away from 0, so if \( a \neq 0 \), \( \lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2} \).
- When \( x \) approaches 0, \( f(x) \) grows unboundedly in the positive direction.
- Use notation \( \lim_{x \to 0} \frac{1}{x^2} = \infty \) to indicate that \( f(x) \) can be made arbitrarily large when \( x \) is sufficiently close to 0.
- Call the vertical line defined by \( x = 0 \) (i.e. \( y \)-axis) a vertical asymptote, to indicate that the graph can be arbitrarily close to the line as it grows.
- We can also consider one-sided limits and we still have \( \lim_{x \to 0^+} \frac{1}{x^2} = \infty \) and \( \lim_{x \to 0^-} \frac{1}{x^2} = \infty \).

Example 2. Let \( f(x) = \frac{1}{x} \). What happens when \( x \to 0 \)?

You should be able to observe the following from the graph:

- When \( x \to 0^+ \), the function grows unboundedly towards the positive direction. In other words, the value of the function is “very large positive”. We can say \( \lim_{x \to 0^+} \frac{1}{x} = \infty \).
• When \( x \to 0^- \), the function grows unboundedly towards the negative direction. In other words, the value of the function is “very large negative”. (Here the word “large” really means the magnitude is large.) We can say \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \).

• If we do not specify in which direction \( x \) approaches 0, since the two one-sided limits do not agree (i.e. the function does not grow in the same direction), we cannot say more than \( \lim_{x \to 0} \frac{1}{x} \) does not exist.

• \( x = 0 \) (i.e. \( y \)-axis) is still a vertical asymptote.

**Example 3.** Let \( f(x) = \ln x \). Consider the behavior of the function when \( x \to 0^+ \).

The following can be observed from the graph:

• When \( x \to 0^+ \), \( \ln x \) is very large negative, so \( \lim_{x \to 0^+} \ln x = -\infty \).

• No left-side limit since the function is not defined when \( x < 0 \).

• Line \( x = 0 \) is still a vertical asymptote. In general a one-sided infinite limit will suffice to conclude the existence of a vertical asymptote.

**Summary**

We use infinite limits denoted by

\[
\lim_{x \to a} = \infty \text{ or } -\infty
\]

to indicate unbounded growth of the function in positive or negative direction, respectively. Similarly, we can define the infinite one-sided limits.

We say a vertical line \( x = a \) is a **vertical asymptote** if at least one of \( \lim_{x \to a} f(x) \), \( \lim_{x \to a^+} f(x) \), and \( \lim_{x \to a^-} f(x) \) is \( \infty \) or \( -\infty \).

**Computation**

The general pattern of the algebra in the computation of infinite limits is as follows: if you have a fraction whose numerator approaches a certain finite non-zero number in a limiting process, while the denominator approaches 0, then the denominator can be realized as a very small positive or negative number. Intuitively, a finite nonzero number divided by a number with very small magnitude will result in a quotient whose magnitude is extremely large, which could be reasonably denoted by “infinity”. The sign of the quotient depends on the signs of both the numerator and the denominator.

**Example 4.** Compute \( \lim_{x \to 0^+} \cot x \).
Solution. It is familiar that \( \cot x = \frac{\cos x}{\sin x} \). So \( \lim_{{x \to 0^+}} \cot x = \lim_{{x \to 0^+}} \frac{\cos x}{\sin x} \). However, when \( x \to 0^+ \), the numerator approaches \( \cos 0 \) which is 1, while the denominator approaches \( \sin 0 \) which is 0. If we use a small positive value of \( x \) to test the sign of the denominator, we know it approaches 0 from the positive side, which means the denominator is a very small positive number in the limiting process. So intuitively the fraction is a very large positive number which concludes that the limit is \( \infty \).

Example 5. Find all vertical asymptotes of the function \( f(x) = \frac{x^2}{x^2 - 1} \).

Solution. We can easily see that, the function \( f(x) \) is continuous away from \(-1\) and 1. So if \( a \) is not \(-1\) or 1, the limit \( \lim_{{x \to a}} \frac{x^2}{x^2 - 1} \) is always the value of the function at \( a \), which is impossible to be unbounded. So the only possible candidates for vertical asymptotes are \( x = -1 \) or \( x = 1 \). You can try to compute the one-sided limits of the function \( f(x) \) at \(-1\) and 1 to conclude that both of them are actually vertical asymptotes. You should be able to finish off the solution yourself.

Remark. In general, in order to find all vertical asymptotes, we only need to look at points at which the function is discontinuous. However, they do not necessarily correspond to vertical asymptote, so you have to check them one by one by computing the (one-sided) limits of the function at these points.

Upcoming

Horizontal asymptotes and limits at infinity with explicit computations, which can be seen as a counterpart of the above, in some sense.