Lecture Outline (Logarithmic Differentiation)  
Monday, March 3  

Announcements  
1. Great job on Midterm 2! You all did very well!  
2. If you want to retake/take the review quiz, you should consider scheduling that soon.  

Motivation  
A few weeks ago we computed the derivative of $\ln(x)$. Today we’ll use this computation together with the rules of logarithms that you should recall from the beginning of the quarter to learn how to differentiate one type of function that you still don’t know how to differentiate, namely the functions of the form $f(x)^{g(x)}$, where $f(x)$ and $g(x)$ are not just constant functions. Some simple examples of functions of this type are $x^x$, $x^{\sin(x)}$, and $\ln(x)^x$.  

Note: When $f(x)$ is a constant function, functions of this form are exponential functions like $e^x$, which you can differentiate. When $g(x)$ is a constant, you get a power function like $x^2$, which you can also differentiate. So you just don’t know how to differentiate $f(x)^{g(x)}$ when both $f(x)$ and $g(x)$ are non-constant functions.  

Logarithmic Differentiation  
Suppose we have a function $f(x)^{g(x)}$ and $f(x)$ and $g(x)$ are non-constant functions. We can’t differentiate $f(x)^{g(x)}$ outright, but we can take advantage of a law of logarithms to do this:  

\[
\begin{align*}
y &= f(x)^{g(x)} \\
\ln y &= \ln(f(x)^{g(x)}) \\
\ln y &= g(x) \ln(f(x))
\end{align*}
\]

Then we can use implicit differentiation to compute $\frac{dy}{dx}$.  

Examples

1. Let \( f(x) = x^x \). Find \( f'(x) \).

2. Let \( g(t) = (t^2 + 2t)^{\sin(t)} \). Find \( g'(t) \).

3. Let \( h(y) = \left(\frac{y}{y^2 + 1}\right)^{\sin^2(y)} \). Find \( h'(y) \).

Note: In general, with implicit differentiation, you answer may involve both \( x \) and \( y \). However, in implicit differentiation, you always konw what \( y \) is as a function of \( x \), and so via substitution, your original answer should always only involve \( x \).

Other uses of Logarithmic Differentiation

It turns out that by taking advantage of logarithmic differentiation and the laws of logarithms, one can find easier ways to differentiate functions that we already know how to differentiate.

Examples

1. Let \( f(x) = (2x - 1)^{10}(3x + 2)^4 \). Find \( f'(x) \).

2. Let \( f(x) = \frac{(2x+1)(x^2+\sin(x))^{3/2}}{\sqrt{x^2+1}} \). Find \( f'(x) \).

Detour: Differentiating with respect to \( x \)

Suppose someone gave you the equation

\[ y = x + t^2 \]

It seems likely that \( y \) is the dependent variable, as usual, but is \( x \) or \( t \) your other variable? If \( x \) is a variable, then \( t \) will be an unknown constant, while if \( t \) is a variable, then \( x \) will be an unknown constant. This makes a difference, since if \( x \) is a variable, then the equation represents a straight line in the \( x - y \) plane with slope 1 and \( y \)-intercept \( t^2 \), while if \( t \) is a variable, the equation represents a parabola in the \( y - t \) plane that opens upward and has vertex at \((0, x)\).

Ideally, the person who gave you the equation would write \( f(x) = x + t^2 \) if it was a function of \( x \), and similairly for \( t \), but sometimes which letter is a variable depends on what
you want to do with your equation. So if you want to view $x$ as a variable, then you will differentiate with respect to $x$, and write

$$\frac{d}{dx}(x + t^2),$$

whereas if you want to view $t$ as a variable, then you will differentiate with respect to $t$, and write

$$\frac{d}{dt}(x + t^2).$$

So when we say "differentiate with respect to $x$", we just mean that $x$ is the variable.