Recap

Last time we developed a number of shortcuts for taking the derivative of $e^x$.

Today we are going to learn how to take the derivative of products and quotients, which will greatly expand our differentiation abilities!!!

The Derivative of a Product

Notice: The limit of the product is not the product of the limits:

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x$$

but

$$\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 \neq 2x.$$

Suppose you wanted to differentiate a function that was a product, say $f(x)g(x)$. For example, how do you differentiate $xe^x$? Using the limit definition of the derivative we can derive:

The Product Rule for derivatives is:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Mantra: "The derivative of the product is the derivative of the first times the second plus the derivative of the second times the first."

Examples

1. Let $y = xe^x$. Find $y'$.
2. Let $y = e^{2x}$. Find $y'$.
3. Let $z = 2xe^x(4e^x - 3\sqrt{x})$. Find $\frac{dz}{dx}$. 

The Quotient Rule

Suppose we have a function that is a quotient: \( h(x) = \frac{f(x)}{g(x)} \). What is \( h'(x) \)? We can derive the Quotient Rule from the Product Rule:

\[
\begin{align*}
  h(x) &= \frac{f(x)}{g(x)} \\
  h(x)g(x) &= f(x) \\
  h'(x)g(x) + g'(x)h(x) &= f'(x) \\
  h'(x)g(x) &= f'(x) - h(x)g'(x) \\
  h'(x) &= \frac{f'(x) - h(x)g'(x)}{g(x)} \\
  &= \frac{f'(x) - \frac{f(x)g'(x)}{g(x)}}{g(x)} \\
  &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{align*}
\]

So the Quotient Rule is:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]

Notice that in the formula above, it is VERY important that the function in the numerator is called \( f(x) \) and the function in the denominator is called \( g(x) \): the minus sign in our expression will give us problems if we switch these names around! Be careful!

Examples

1. Let \( y = \frac{x^2 - 1}{x + 2} \). Find \( y' \).

2. Let \( y = \frac{xe^x}{x+1} \). Find the equation of the tangent line to this curve at the point \((1, e/2)\).

What to Know/Memorize

1. The Product Rule:

\[
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
\]

2. The Quotient Rule:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]