Lecture Outline (Derivative as a Function II)  
Wednesday, January 30

Recap

Last time we talked about the derivative of a function as a function itself. For a function \( f(x) \) we said the derivative can be written either as \( f'(x) \) or \( \frac{d}{dx}[f(x)] \), and is given by

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

We computed the derivative of several of our favorite functions.

Terminology

**Definition:** A function \( f \) is **differentiable at** \( a \) if \( f'(a) \) exists. It is **differentiable on an open interval** \((a, b)\) if it is differentiable at every number in the interval.

For this section, let’s use \( y = f(x) = x^2 \) as a running example. We said that \( f'(x) = 2x \) is called the derivative of \( x^2 \). The process of finding the derivative is called **differentiation**. You might be asked to ”**differentiate** \( x^2 \),” in which case you would write down the derivative of \( x^2 \), which is \( 2x \).

Non-Differentiability

Although we will be dealing mostly with differentiable functions in this class, there are a handful of times when we will encounter functions which have points which are not differentiable.

The function \( f(x) \) is **not differentiable** at \( x = a \) if

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]
does not exist.

There are three typical types of non-differentiability.
1. **Corners** A function is not differentiable at a point where the graph of $f$ has a kink or corner. Essentially, these places fail to be differentiable because the left and right-hand limits do not match up.

2. **Discontinuities** A function is not differentiable at a point where the graph of $f$ is not continuous.

3. **Vertical Tangents** Finally, a function is not differentiable at a point on the graph where the tangent line to $f$ is a vertical line. This is because the slope of the tangent line to the graph at this point is infinite, which in our class means "does not exist."

**Notation**

There are many different notations in use to signify the derivative. If $y = f(x) = x^2$, we can write:

$$2x = f'(x) = y' = (x^2)' = \frac{d}{dx} x^2 = \frac{d(x^2)}{dx} = \frac{dy}{dx}.$$  

The first three in the list should make some sense. But what are the last three? First of all, $\frac{d}{dx}$ is not a fraction (and so you cannot cancel the $d$ in the numerator with the $d$ in the denominator!) You should just regard $\frac{d}{dx}$ as a fixed symbol.

**More Derivatives**

We started with a function $f(x)$ and differentiated to get a new function $f'(x)$. But then we could differentiate $f'(x)$ to get another new function! We need better terminology. So call $f'(x)$ the **first derivative** of $f(x)$. The the derivative of $f'(x)$ is written $f''(x)$ and is called the **second derivative** of $f(x)$. But of course $f''(x)$ is also the (first) derivative of $f'(x)$. If we wanted to differentiate again, we would get $f'''(x)$. This is the third derivative of $f(x)$, the second derivative of $f'(x)$, and the (first) derivative of $f''(x)$. In general, the $n$-th derivative of $f(x)$ is written $f^{(n)}(x)$.

If we had written $y = x^2$, then the first derivative would be $y' = 2x$, and the second derivative would be $y'' = 2$, and the third derivative would be $y''' = 0$. 

2