Construction of a measure on the group of (increasing) homeomorphisms of [0,1], quasi-invariant under C^α diffeomorphisms for any $\alpha > 3/2$.

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We describe a simple process to choose a random monotone increasing function on [0,1], that is a mapping from a probability space into the group G of homeomorphisms of [0,1]. The image of the probability measure under this map is a measure on the group. We shall then show that the measure thus obtained is quasi-invariant under the action of C^α diffeomorphisms on the (range side of the) group.

Choose $f(1/2)$ with uniform distribution on [0,1]. Next, choose $f(1/4)$ with uniform distribution on $[0, f(1/2)]$ and, independently, $f(3/4)$ with uniform distribution on $[f(1/2), 1]$.

After n steps we have chosen the values $f(j2^{-n})$ for $j = 1, 2, \dots, 2^n - 1$, and the next step consists in choosing, independently for the different values of j, the value of $f((2j-1)2^{-n-1})$ in the interval $I_{n,j} = [f((j-1)2^{-n}), f(j2^{-n})]$,with uniform distribution.

We write $a_{n,j} = f(j2^{-n}) - f((j-1)2^{-n})$. Clearly: $a_{n,j} = a_{n+1,2j-1} + a_{n+1,2j}$, and the expectation

$$E(a^2_{n+1,2j-1} + a^2_{n+1,2j}) = (a^2_{n,j} \int x^2 + (1-x)^2 dx = (2/3)a^2_{n,j}$$

which implies

$$E(\sum_j a^2_{n,j}) = (2/3)^n \tag{1}$$

It follows from (1) that $\max_j a_{n,j} \to 0$ a.s. as $n \to \infty$ and the function f can therefore be extended to a continuous, (actually Holder), strictly increasing mapping of [0,1] onto itself, that is, an element of G. We denote by μ the image of the probability measure under this process.

Let F be a C^2 diffeomorphism of [0,1], $F(0) = 0, F(1) = 1$. F acts on G by left multiplication (composition), and hence on the measures on G. We denote $\nu = F(\mu)$. ν can be described explicitely as follows: $F \circ f(x) < \lambda \Leftrightarrow f(x) < F^{-1}(\lambda)$ hence the density of distribution of $F \circ f(1/2)$ is $\Phi(x)dx$ where $\Phi = d/dxF^{-1}$. Similarly, $F \circ f(1/4)$ is distributed in $[0, F \circ f(1/2)]$ with density proportional to $\Phi(x)dx$, etc. Thus we can obtain ν directly by repeating the procedure that gave us μ except that instead of the uniform density on each interval we put the density proportional to Φdx.

The Radon-Nikodym derivative $d\nu/d\mu$ is given explicitely as an infinite product: it's value at a "point" f is

$$d\nu/d\mu\,(f) = \prod_n \prod_{j=1}^{2^{n-1}} \Phi_n(f(\frac{2j-1}{2^n})) \tag{2}$$

where $\Phi_1 = \Phi,\ \Phi_n(x) = c_{n,j}\Phi(x)$ on $I_{n-1,j}$ and the constants $c_{n,j}$ are such that

$$\int_{I_{n-1,j}} c_{n,j}\Phi(x)dx = f(j2^{1-n}) - f((j-1)2^{1-n}) = a_{n-1,j} \tag{3}$$

and we want to show that the product (2) converges a.s.

Conditioned on $\{f(j2^{1-n})\}_{j=1}^{2^{n-1}}$ the product $\prod_{j=1}^{2^{n-1}} \Phi_n(f(\frac{2j-1}{2^n}))$ has the form $\prod_{j=1}^{2^{n-1}} (1 + y_{n,j})$ with $y_{n,j}$ independent for $j = 1, \cdots, 2^{n-1}$,and

$$E(y_{n,j}) = 0 \quad and \quad \mid y_{n,j} \mid < Const\, a_{n-1,j} \tag{4}$$

(this last estimate uses the fact that $\Phi \in C^1$ and is bounded away from zero; if we assume only $F \in C^{1+\eta}$ we obtain the estimate $\mid y_{n,j} \mid < Const\, a_{n-1,j}^{\eta}$ and things work for $\eta > 1/2$ and break for $\eta = 1/2$ where I believe I have an example of $\nu \perp \mu$). By (4),

$$E((\sum_j y_{n,j})^2 \mid \{f(j2^{1-n})\}_{j=1}^{2^{n-1}}) < const \sum_j a_{n-1,j}^2 \tag{5}$$

and by (1) this is bounded by $n^2(2/3)^{n-1}$ on all but n^{-2} of the space of choices of $\{f(j2^{1-n})\}_{j=1}^{2^{n-1}}$. This gives

$$\sum P(\mid \sum y_{n,j} \mid > n^6(2/3)^n) < \infty$$

and (2) converges a.s.