1. (Quickies)

a. Let \((B, ||\cdot||)\) be a normed space, and \(A : B \to B\) an invertible linear transformation such that \(\|A^n\| \leq c\) for some constant \(c > 0\) and all \(n \in \mathbb{Z}\). Prove that there is an equivalent norm on \(B\) with respect to which \(A\) is an isometry.

**Answer:** For \(x \in B\) set
\[
\|x\|_* = \limsup_{N} \frac{1}{2N} \sum_{-N}^{N} \|A^n x\|.
\]

b. Let \(\mu\) be a finite measure on \([-1, 1]\) and assume that \(\int x^{kn} = 0\) for some integer \(k\) and all nonnegative integers \(n\). Prove that if \(k\) is odd then \(\mu = 0\). What can you say when \(k\) is even?

**Answer:** If \(k\) is odd, the polynomials in \(x^k\) are dense in \(C([-1, 1])\).

If \(k\) is even, the polynomials in \(x^k\) are dense in the subspace of even functions in \(C([-1, 1])\), and hence, for all \(f \in C([-1, 1])\),
\[
\int (f(x) + f(-x)) d\mu = \int f(x) d(\mu(x) + \mu(-x)) = 0
\]
which is equivalent to \(\mu(x) + \mu(-x) = 0\).

c. Prove that the space \(C(0, 1)\) is not reflexive.

**Hint:** Identify the dual space \(M(0, 1)\) of \(C(0, 1)\) and show that the dual of \(M(0, 1)\) contains elements that can not be identified with elements of \(C(0, 1)\).

**Answer:** The dual space \(M(0, 1)\) is the space of finite Borel measures (the Riesz representation theorem).

The map \(\Phi : \mu \mapsto \mu([0, \frac{1}{2}])\) is in the dual space of \(M(0, 1)\).

To see that there is no continuous function \(f\) such that \(\langle \Phi, \mu \rangle = \inf f d\mu\), denote by \(\delta_t\) the Dirac measure at \(t\) and observe that \(\langle \Phi, \delta_t \rangle = 1_{[0,\frac{1}{2}]}(t)\), while \(\int f d\delta_t = f(t)\) is continuous.

d. Prove that \(C(0, 1)\) is not isomorphic—and in particular not isometric—to a uniformly convex Banach space.

**Answer:** Uniformly convex Banach spaces are reflexive, and reflexivity is preserved under isomorphism. \(C(0, 1)\) is not reflexive.
2 Prove that a linear operator $T$ on a Hilbert space $\mathcal{H}$ is compact if, and only if, it is the limit, in the norm topology of operators, of a sequence of operators of finite rank.

**Answer:** By definition, $T$ is compact if $TB(0, 1)$, the $T$ image of the unit ball $B(0, 1)$, is precompact (has compact closure), or equivalently, totally bounded.

Assume $T$ compact. Let $\varepsilon > 0$. Since $TB(0, 1)$ is totally bounded cover it by a finite number of balls of radius $\varepsilon$, and let $V_\varepsilon$ be the subspace spanned by the centers of these balls. Let $\pi_V$ be the orthogonal projection of $\mathcal{H}$ onto $V$. Then $\pi_V T$ is of finite rank and, since $Tv$ is within $\varepsilon$ from $V$ for every $v \in B(0, 1)$, we have $\|T - \pi_V T\| \leq \varepsilon$.

Conversely, assume that for some $\varepsilon > 0$, there exists an operator $S$ of finite rank such that $\|T - S\| < \varepsilon / 2$. Since $SB(0, 1)$ is a bounded set in a finite dimensional space, it can be covered by a finite number of balls of radius $\varepsilon / 2$, and the concentric balls of radius $\varepsilon$ cover $TB(0, 1)$. If such $S$ can be found for every $\varepsilon > 0$ then $TB(0, 1)$ is totally bounded.
Suppose $B$ is a Banach space and $K \subset B$ is a subset. Recall that its convex hull, $\text{ch}(K)$, is the smallest convex subset of $B$ which contains $K$.

**a.** Prove that if $K$ is compact, then the closure $\overline{\text{ch}(K)}$ of $\text{ch}(K)$ is compact as well.

**Answer:** Since $\overline{\text{ch}(K)}$ is closed, it is enough to show that it is totally bounded. $\text{ch}(K)$ is the set of all finite convex combinations of elements of $K$.

The assumption that $K$ is compact guarantees that given $\varepsilon > 0$ there exists a finite subset $K_\varepsilon \subset K$ which is $\varepsilon/2$ dense in $K$. The convex combinations of elements of $K_\varepsilon$ are $\varepsilon/2$ dense in $\text{ch}(K)$. If $N > \frac{2 \max_{v \in K} \|v\|}{\varepsilon}$, the (finite) set of convex combinations of elements of $K_\varepsilon$ with coefficients of the form $\frac{j}{N}$ is $\varepsilon$-dense in $\overline{\text{ch}(K)}$.

**b.** Show that the set of indicator functions $A = \{1_{[\tau, \tau+\frac{1}{5}]}(t) : \tau \in \mathbb{T}\}$ is compact in $L^1(\mathbb{T})$, and its convex hull is not.

**Answer:** $A$ is the range of the continuous map $\tau \mapsto 1_{[\tau, \tau+\frac{1}{5}]}$ of the compact space $\mathbb{T}$ into $L^1(\mathbb{T})$. The convex hull of $A$ contains no continuous functions. Its closure contains all the convolutions $\varphi \ast 1_{[0, \frac{1}{5}]}$ where $\varphi$ is non-negative, continuous, and of integral 1. All these are continuous.
Let $A_j \subset [0, 1]$, for $j = 1, 2, \ldots, N$ be Lebesgue measurable, $\mu(A_j) \geq \frac{1}{2}$.

Let $0 < a < 1/2$ and denote $E_a = \{x : x \in A_j \text{ for more than } aN \text{ values of } j\}$.

Prove that $\mu(E_a) \geq \frac{1-2a}{2(1-a)}$.

Show that the estimate $\mu(E_a) \geq \frac{1-2a}{2(1-a)}$ is "best possible" (if it is to apply to all $N$).

**Hint:** Consider $F = \sum_1^N 1_{A_j}$.

**Answer:** Write $F = \sum_1^N 1_{A_j}$. Then

$$\frac{1}{2}N \leq \int F(x) \, dx \leq \int_{E_a} F(x) \, dx + aN(1 - \mu(E_a)) \leq N\mu(E_a) + aN(1 - \mu(E_a)) \quad (1)$$

and $N(1 - a)\mu(E_a) \geq N^{1-2a}$.

To show that the estimate can not be improved, given $0 < a < 1/2$, define $A_j$ so that the inequalities in (1) are close to equalities.

All the sets will contain $I_{a,\varepsilon} = [0, \frac{1-2a}{2(1-a)} + \varepsilon]$, where $\varepsilon = \varepsilon_N \to 0$ as $N \to \infty$—so that on $I_{a,\varepsilon}$ one has $F(x) = N$—while the sets $A_j \cap \left[\frac{1-2a}{2(1-a)} + \varepsilon, 1\right]$ are chosen in a way to have even covering—$F$ is essentially a constant ($< aN$).
The Hausdorff-Young inequality on the line states that if $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then

$$f \in L^p(\mathbb{R}) \quad \text{implies} \quad \|\hat{f}\|_{L^q(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}. \quad (2)$$

Prove that the converse is true: (2) implies $1/p + 1/q = 1$ and $1 \leq p \leq 2$.

**Hint:** For the first claim use scaling ($f_{\lambda,s} = \lambda^s f(\lambda x)$ for appropriate $s$, and its effect on the Fourier transform). For the second claim, show that if $\varphi(x) = e^{i\lambda x^2} 1_{[-1,1]}$, then as $\lambda \to \infty \|\hat{\varphi}_\lambda\|_{\infty} \to 0$ while $\|\hat{\varphi}_\lambda\|_2$ remains constant.

**ANSWER:** The map $f \mapsto f_{\lambda,p^{-1}}$ is an isometry on $L^p(\mathbb{R})$.

Let $f \in L^p(\mathbb{R})$. The Fourier transform of $f_{\lambda,p^{-1}}$ is

$$\hat{f}_{\lambda,p^{-1}}(\xi) = \lambda^{\frac{1}{p}-1} \int f(\lambda x) e^{i\xi x} d\lambda x = \lambda^{\frac{1}{p}-1 + \frac{1}{q}} \lambda^{\frac{1}{q}} \hat{f}(\xi) = \lambda^{\frac{1}{p}-1+\frac{1}{q}} \hat{f}_{\lambda^{-1},q}(\xi).$$

If $\frac{1}{p} - 1 + \frac{1}{q} \neq 0$ one can choose $\lambda$ (close to 0 if $\frac{1}{p} - 1 + \frac{1}{q} < 0$, large if $\frac{1}{p} - 1 + \frac{1}{q} > 0$) to make $\lambda^{\frac{1}{p}-1+\frac{1}{q}}$ arbitrarily big.

For the second claim: van der Corput’s lemma gives $\|\hat{\varphi}_\lambda\|_{\infty} = O^*(\lambda^{-\frac{1}{2}})$. Since $\|\hat{\varphi}_\lambda\|_2$ remains constant, $\int |\hat{\varphi}_\lambda(\xi)|^q d\xi = O\left(\lambda^{-\frac{q-2}{2}}\right)$. This shows that if $q > 2$ the map $f \mapsto \hat{f}$ of $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ is not bounded below, and the inverse Fourier transform, which is essentially the Fourier transform, cannot be defined as a bounded operator $L^q(\mathbb{R}) \to L^p(\mathbb{R})$. 
6  (Quickies)

a. A distribution \( \mu \) on \( \mathbb{T} \) is positive if \( \langle f, \mu \rangle \geq 0 \) for every nonnegative \( f \in C^\infty(\mathbb{T}) \).

Show that a positive distribution is (defined by) a measure.

Hint: Positivity implies: for real-valued \( C^\infty \) functions \( f \), \( \langle f, \mu \rangle \leq \max f(t) \langle 1, \mu \rangle \).

**Answer:** Use the hint for both \( f \) and \(-f\) and conclude that for real-valued \( f \in C^\infty(\mathbb{T}) \),
\[
|\langle f, \mu \rangle| \leq K \| f(t) \|_\infty
\]
with \( K = \langle 1, \mu \rangle \). Since \( C^\infty(\mathbb{T}) \) is dense in \( C(\mathbb{T}) \), \( \mu \) has a unique extension to a bounded linear functional on \( C(\mathbb{T}) \). By the Riesz representation theorem it is a measure on \( \mathbb{T} \).

b. Assume \( f_n \in L^2[0,1], n \in \mathbb{N}, \) and \( \| f_n \| \leq 1 \).

Prove that \( \mu(\{ x \mid |f_n(x)| > n^{\frac{3}{2}} \}) < n^{-\frac{2}{3}}, \) and conclude that for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that the measure of the set \( A_N = \{ x : |f_n(x)| \leq n^{\frac{3}{2}} \} \) for all \( n > N \) exceeds \( 1 - \varepsilon \).

**Answer:** The first estimate is Chebyshev’s (or weak type) inequality. The second claim follows from
\[
\mathbb{T} \setminus A_N = \bigcup_{n>N} \{ x : |f_n(x)| > n^{\frac{3}{2}} \}
\]
and its measure is bounded by \( \sum_{n>N} n^{-\frac{4}{3}} \sim N^{-\frac{1}{3}} \).

7  Let \{\( f_n \)\} be an orthonormal sequence in \( L^2(0,1) \).

Prove that \( S_n = \frac{1}{n} \sum f_m \to 0 \) a.e.

**Hints:**

a. \( \| S_n \|_{L^2} = \frac{1}{n} \). It follows that if \( \sum \lambda_j^{-1} < \infty \), and in particular if \( \lambda_j = [j \log^2 j] \), then \( \sum |S_{\lambda_j}|^2 \) converges a.e. and \( S_{\lambda_j} \to 0 \) a.e.

b. If \( N \in (\lambda_j, \lambda_{j+1}) \), then \( S_N = \frac{\lambda_j}{N} S_{\lambda_j} + \frac{1}{N} \sum_{\lambda_j} S_{\lambda_j+1} f_m \). Use 6.b. to estimate the last sum.
Let $B = \{(x, y, z) : r = \sqrt{x^2 + y^2 + z^2} \leq 1\}$, the unit ball in $\mathbb{R}^3$, $G = \mathbb{1}_B$ its indicator function, and $g(x) = \int G(x, y, z) \, dy \, dz$.

a. Compute $g$.

b. Show that $\hat{g}(\xi) = O\left(|\xi|^{-3}\right)$

c. Prove $\hat{G}(\xi, \eta, \zeta) = O\left((\xi^2 + \eta^2 + \zeta^2)^{-\frac{3}{2}}\right)$.

d. Let $F_n(x, y, z) = \sin^2(nr)G(x, y, z)$, and $f_n(x) = \int F_n(x, y, z) \, dy \, dz$.

Prove: $f_n \to \frac{1}{2}g$ uniformly as $n \to \infty$.

**Answer:**

a. $g(x) = \pi(1 - x^2)$ for $|x| \leq 1$, zero elsewhere.

b. Use the fact: $g'' = -2\pi \mathbb{1}_{[-1,1]}$.

c. $\hat{G}$ is radial and $\hat{G} = \hat{g}$ on the real axis.

d. The simplest form of van der Corput gives rate $O\left(\frac{1}{n}\right)$.
Let \( f(x) = \sum 10^{-n} \cos 10^{2n}x \).

a. Prove that \( f \) satisfies the Hölder \( \frac{1}{2} \) condition:

\[
|f(x + h) - f(x)| \leq \text{const} \cdot h^{\frac{1}{2}}.
\]

**ANSWER:** For \( m \in \mathbb{N} \) write

\[
S_m(x) = \sum_{n<m} 10^{-n} \cos 10^{2n}x,
\]

and \( T_m(x) = \sum_{n>m} 10^{-n} \cos 10^{2n}x. \)

Given \( h \), let \( m \) be such that \( h \sim 10^{-2m} \) and write

\[
f(x) = S_m(x) + 10^{-m} \cos 10^{2m}x + T_m(x).
\]

Observe that \( \| \frac{d}{dx} S_m \|_\infty \leq 1.2 \cdot 10^{m-1} \) and \( \| T_m \|_\infty \leq 1.2 \cdot 10^{-m-1} \) so that

\[
|f(x + h) - f(x)| \leq |S_m(x + h) - S_m(x)| + 10^{-m} |\cos 10^{2m}(x + h) - \cos 10^{2m}x| + |T_m(x + h) - T_m(x)|.
\]

We have \( |S_m(x + h) - S_m(x)| \leq |h| \| \frac{d}{dx} S_m \|_\infty \sim 10^{-m} \sim |h|^{\frac{1}{2}} \) and the other two terms are bounded uniformly by \( 10^{-m} \sim |h|^{\frac{1}{2}}. \)

b. Prove that \( f \) is nowhere differentiable.

**Hint:** For every \( x \) and every \( n \) find points \( y_n \) and \( z_n \) such that

\[
|x - y_n| \sim 10^{-2n}, \quad |x - z_n| \sim 10^{-2n}, \quad \frac{f(x) - f(y_n)}{x - y_n} > 10^{n-2}, \quad \text{and} \quad \frac{f(x) - f(z_n)}{x - z_n} < -10^{n-2}.
\]

**ANSWER:** Given \( x \) an \( n \), if \( \cos 10^{2n}x > 0 \) let \( y_n \) be the point closest to \( x \) on the left such that \( \cos 10^{2n}y_n = -1 \) and \( z_n \) the closest on the right satisfying the same condition. If \( \cos 10^{2n}x \leq 0 \) define \( y_n \) and \( z_n \) as the closest neighbors on either side on which \( \cos 10^{2n}y_n = \cos 10^{2n}z_n = 1 \).

Since \( \cos 10^{2n}x \) is \( 2\pi 10^{-2n} \) periodic, \( \frac{\pi}{2} 10^{-2n} \leq |x - y_n| \leq 2\pi 10^{-2n} \) and similarly for \( z_n \).

Thus,

\[
\frac{10^{-n} \cos 10^{2n}x - 10^{-n} \cos 10^{2n}y_n}{x - y_n} \geq 2 \pi 10^{2n} 10^{-n} = \frac{2}{\pi} 10^n \geq 0.5 10^n.
\]

Now write again

\[
f(x) = S_n(x) + 10^{-n} \cos 10^{2n}x + T_n(x).
\]

and observe that

\[
\left| \frac{f(x) - f(y_n)}{x - y_n} - 10^{-n} \frac{\cos 10^{2n}x - \cos 10^{2n}y_n}{x - y_n} \right| \leq \left\| \frac{d}{dx} S_n \right\|_\infty + \frac{4}{\pi} 10^{2n} \| T_n \|_\infty \leq 0.3 10^n.
\]
It follows that
\[
\left| \frac{f(x) - f(y_n)}{x - y_n} \right| > 0.2 \, 10^n. \tag{6}
\]
We obtain the same estimate, with reversed sign, for \( z_n \).

c. Can a Lipschitz function be nowhere differentiable? (Justify your answer by quoting relevant standard theorems.)

**ANSWER:** No! a Lipschitz function is of bounded variation and, by Lebesgue’s theorem, is differentiable a.e.
Let $B$ be a Banach space and $S$ a linear map from $B$ into $C([0,1])$, such that if $\{v_n\} \subset B$ and $\|v_n\|_B \to 0$ then $Sv_n(x) \to 0$ pointwise in $[0,1]$. Prove that $S$ is bounded; in particular, the assumptions $\|v_n\|_B \to 0$ implies $Sv_n(x) \to 0$ uniformly.

**Answer:** For every $x \in [0,1]$ the map $\varphi_x : v \mapsto Sv(x)$ is a bounded linear functional on $B$. The problem is to show that these functionals are uniformly bounded.

Assume they are not. Let $\{x_n\}$ be such that $\|\varphi_{x_1}\| > 1$, and $\|\varphi_{x_n}\| > 10^n\|\varphi_{x_{n-1}}\|$ for all $n > 1$. Let $v_n \in B$ such that $\|v_n\| = 5^{-n}\|\varphi_{x_{n-1}}\|$, and $Sv_n(x_n) > 2^n$. Write $v = \sum v_n$. Since $S(v - \sum_{n=1}^m v_n)$ converges to zero pointwise, $Sv = \sum Sv_n$ which is not bounded.