

## Characterization of $C(\mathfrak{M})$

by

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Consider a Banach algebra  $B$  of continuous functions on a compact Hausdorff space  $\mathfrak{M}$ . The purpose of this note is to prove the following theorem which generalizes the main result of [2]:

**THEOREM.** *Assume that for every closed subset  $F$  of  $\mathfrak{M}$ , there exists a positive number  $\varepsilon = \varepsilon(F)$  such that whenever  $N$  is both closed and open in  $F$ ,  $B$  contains an element  $h$  of norm one satisfying  $\operatorname{Re}(h(M)) < 0$  for  $M \in N$ ,  $\operatorname{Re}(h(M)) > \varepsilon$  for  $M \in F - N$ . Then  $B = C(\mathfrak{M})$ .*

The main difference between the present result and that of [2] is that we discard the assumption that  $B$  is self adjoint and regular, and that instead of assuming the boundedness of idempotents (which we do not know to exist *a priori*) we assume a uniform separation of parts of  $F$  by elements of  $B$  of norm one for every closed  $F \in \mathfrak{M}$ .

If in particular  $B$  is regular with a unit and in every quotient algebra of  $B$  the idempotents are bounded— $B = C(\mathfrak{M})$ . This also implies a theorem of Glickberg's [1].

The proof runs very much along the lines of [2] replacing regularity by “almost regularity” in the following sense:

**LEMMA 1.** *For any pair of disjoint closed sets  $N_1, N_2 \subseteq \mathfrak{M}$  and  $\varepsilon > 0$ , there exists  $h \in B$  satisfying  $|h(M)| < \varepsilon$  on  $N_1$ ,  $|h(M) - 1| < \varepsilon$  on  $N_2$ .*

We shall call  $h$  an  $\varepsilon$ -idempotent with respect to  $(N_1, N_2)$ .

*Proof.* According to the assumption of the theorem there exists an  $\varepsilon_1 > 0$  and  $h_1 \in B$ ,  $\|h_1\| = 1$ , such that  $\operatorname{Re}(h_1(M)) < 0$  on  $N_1$  and  $\operatorname{Re}(h_1(M)) > \varepsilon_1$  on  $N_2$ . let  $P(z)$  be a polynomial satisfying:  $P(0) = 0$  and

$$\begin{aligned} |P(z)| &< \varepsilon && \text{for } |z| \leq 1 \text{ and } \operatorname{Re}(z) \leq 0, \\ |P(z) - 1| &< \varepsilon && \text{for } |z| \leq 1 \text{ and } \operatorname{Re}(z) \geq \varepsilon_1 \end{aligned}$$

We can take  $h(M) = P(h_1(M))$ .

It is easily checked that “almost regularity” can replace regularity in proving an analogue of lemma 3 in [2] and in order to conclude the proof following the line of thought presented in [2] we have only to prove the following

**LEMMA 2.** *Assume that for a fixed positive  $\varepsilon_0$ , and every pair of disjoint closed sets  $N_1, N_2 \subseteq \mathfrak{M}$ , an element  $h \in B$  can be found satisfying  $\|h\| = 1$ ,  $\operatorname{Re}(h(M)) < 0$*

on  $N_1$ ,  $\operatorname{Re}(h(M)) > \varepsilon$  on  $N_2$ . Then  $B = C(\mathfrak{M})$  and  $\|f\|_B \leq K_0 \sup |f|$  where  $K_0$  depends only on  $\varepsilon$ .

The proof will be done in three steps.

*Step 1.* For  $\eta > 0$  there exists a finite  $K = K\eta$ , depending on  $\eta$  and  $\varepsilon_0$  alone, such that for every pair of closed disjoint sets  $N_1, N_2 \subset \mathfrak{M}$   $B$  contains an  $\eta$ -idempotent with respect to  $(N_1, N_2)$  of norm less than  $K\eta$ .

This is merely a restatement of lemma 1 with the additional remark that  $\|h\|$  is bounded by the sum of the absolute values of the coefficient of  $P(z)$  which depends only on  $\eta$  and  $\varepsilon_0$ .

*Step 2.* For every  $\eta > 0$  there exists a finite  $K' = K'(\eta)$  such that for every pair of closed disjoint sets  $N_1, N_2$  in  $M$ ,  $B$  contains an  $\eta$ -idempotent with respect to  $(N_1, N_2)$ , of norm less than  $K'(\eta)$ , and whose spectrum lies in the band  $\operatorname{Im}(z) < \eta$ .

*Proof.* Let  $1 > \eta > 0$  be given; put  $\eta_1 = \frac{\eta}{4}$ ,  $\eta_2 = \frac{\eta_1}{2K_{\eta_1}}$ . Let  $N_1, N_2$  be disjoint and closed in  $\mathfrak{M}$  and let  $h(M)$  be an  $\eta_1$ -idempotent with respect to  $(N_1, N_2)$  of norm  $\leq K_{\eta_1}$ , assuming  $\max \operatorname{Im}(h(M)) > \eta$  we shall show how  $h(M)$  can be changed so as to fulfill all the above mentioned requirements.

Denote

$$P_1 = \left\{ M; \operatorname{Im}(h(M)) \leq \frac{1}{4} \max \operatorname{Im}(h) \right\}$$

$$P_2 = \left\{ M; \operatorname{Im}(h(M)) \geq \frac{1}{2} \max \operatorname{Im}(h) \right\}$$

and let  $h^*$  be an  $\eta_2$ -idempotent with respect to  $(P_1, P_2)$   $\|h^*\| < K_{\eta_2}$ .

Set  $h_1 = h - i \frac{\max \operatorname{Im}(h)}{4K_{\eta_2}} h^*$

we have  $\|h_1\| \leq K_{\eta_1} + \frac{\max \operatorname{Im}(h)}{4}$

$$\max \operatorname{Im}(h_2) \leq \left( 1 - \frac{1 - \eta_2}{4K_{\eta_2}} \right) \max \operatorname{Im}(h)$$

$|h_1(M)| < \eta_1 + \eta_2 \frac{\max \operatorname{Im}(h)}{4K_{\eta_2}}$  on  $N_1$

$|h_1(M) - 1| < \eta_1 + \eta_2 \frac{\max \operatorname{Im}(h)}{4K_{\eta_2}}$  on  $N_2$

and  $\min h_1 \geq \min h - \eta_2 \frac{\max \operatorname{Im}(h)}{4K_{\eta_2}}$ .

We can now proceed by induction defining  $h_j^*$  after knowing  $h_j$  in the same way  $h^*$  was obtained from  $h$  and then putting  $h_{j+1} = h_j - i \frac{\max \operatorname{Im}(h_j)}{4K_{\eta_2}} h_j^*$ .

We have

$$\max \operatorname{Im}(h_n) \leq \left( 1 - \frac{1 - \eta_2}{4K_{\eta_2}} \right)^n \max \operatorname{Im}(h) \leq \left( 1 - \frac{1 - \eta_2}{4K_{\eta_2}} \right)^n K_{\eta_1}$$

$$\|h_n\| \leq K_{\eta_1} + \frac{\max \operatorname{Im}(h)}{4} + \sum_{j=1}^{n-1} \frac{\max \operatorname{Im}(h_j)}{4} \leq K_{\eta_1} \left( 1 + \frac{K_{\eta_2}}{1 - \eta_2} \right)$$

$$|h_2(M)| \leq \eta_1 + \eta_2 \frac{\max \operatorname{Im}(h)}{4K_{\eta_2}} \cdot \frac{4K_{\eta_2}}{1 - \eta_2} < 2\eta_1 \text{ on } N_1$$

and similarly

$$\begin{aligned} |h_n(M) - 1| &< 2\eta_1 \text{ on } N_2 \\ \min h_n &> \min h - \eta_1 \end{aligned}$$

For  $n_0$  big enough  $h_{n_0}$  is a  $2\eta_1$ -idempotent with respect to  $(N_1, N_2)$  satisfying the additional requirement  $\max \text{Im}(h_{n_0}) < \eta_1$ . We now repeat the same argument on  $h_{n_0}$  cutting its spectrum on the lower half plane changing its values on  $N_1$  and  $N_2$  by  $\eta_1$  at most, raising  $\max \text{Im}(h_n)$  by  $\eta_1$  at most and its norm by  $\frac{4K\eta_1 K\eta_2}{1-\eta_2}$  and step 2 is proved.

*Step 3:* We notice first that the following is true (cf. [2] lemma 1): Suppose that there exist constants  $K$  and  $K_1 < 1$  such that to any function  $f \in C(\mathfrak{M})$  there exists an element  $f_1 \in B$  such that  $\|f_1\| < K \sup |f(M)|$  and such that  $\sup |f - f_1| < K_1 \sup |f|$ ; then  $B = C(\mathfrak{M})$  and  $\|f\|_B \leq K(1 - K_1)^{-1} \sup |f|$ .

We fix a sufficiently small  $\eta$ . Let  $f \in C(\mathfrak{M})$  repeating for  $f$  the argument of step 2 using  $\eta$ -idempotent with spectrum in the band  $|\text{Im}(z)| < \eta$  and norms bounded by  $K'\eta$ , we see that an element  $g_1 \in B$  exists satisfying

$$\begin{aligned} \|g_1\| &\leq \frac{K\eta}{1-\eta} \sup |f| \\ \max \text{Im}(f - g_1) &< \eta \\ \min \text{Im}(f - g_1) &> -(1 + \eta) \sup |f| \\ |\text{Re}(f - g_1)| &< (1 + \eta) \sup |f| \end{aligned}$$

By the same argument we find an element  $g_2 \in B$  having norm bounded by  $(1 + \eta) \frac{K\eta}{1-\eta} \sup |f|$  and such that

$$\begin{aligned} |\text{Im}(f - g_1 - g_2)| &< \eta + 2\eta \sup |f| \\ \text{and } |\text{Re}(f - g_1 - g_2)| &< (1 + \eta)^2 \sup |f| \end{aligned}$$

We then proceed to “cut” the spectrum on the right half plane and on the left half plane without changing its being bounded in the strip  $|\text{Im}(z)| < 10\eta \sup |f|$ . We thus find an element  $f_1 \in B$  of norm less than  $C_\eta K'_\eta \sup |f|$  such that both  $|\text{Re}(f - f_1)| < 10\eta \sup |f|$  and  $|\text{Im}(f - f_1)| < 10\eta \sup |f|$  hence  $\sup |f - f_1| < 15\eta \sup |f|$  and the theorem is proved.

### References

- [1] I. Glicksberg, *Function algebras with closed restrictions*, to appear.
- [2] Y. Katznelson, *A characterization of the algebra of all continuous functions on a compact, Hausdorff space*, B.A.M.S. **66** 1960, 313–315.