The action of diffeomorphism of the circle on the Lebesgue measure

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This paper continues, and in some sense completes, the study begun in [2]. We consider diffeomorphisms f of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with irrational rotation number. A theorem of Denjoy states that if $f \in C^2$, it is conjugate to a rotation. The conjugating homeomorphims may be non-absolutely-continuous in which case the unique f-invariant measure on $\mathbb{T}$ is not equivalent to the Lebesgue measure μ. On the other hand, it is clear that f maps μ onto a measure equivalentto it, and it is this non-singular transformation that we proposie to study here.

The natural classification theory for non-singular transformations on a Lebesgue space, which parallels the classification of von Neumann factors, is done modulo orbit-equivalence (also know as Dye-equivalence or weak- equivalence). We list the main definitions and results concerning orbit- equivalence, which we use, in section 1. The main contributors to the theory of orbit-equivalence are Hopf, Dye, Krieger, Connes and Woods, and we refer the reader to their works or to the forthcoming exposition [3] for the proofs.

In section 2 we give a characterization of those transformationswhich are orbit-equivalent to smooth diffeomorphisms of the circle. We show that every C^2-diffeomorphism with irrational rotation number is *of product type*, and that every transformation of product type is orbit-equivalent to some C^∞-diffeomorphism.

Using the same ideas, one can easily show that *any* non-singular ergodic system is orbit-equivalent to some *homeomorphism* of $\mathbb{T}$ acting on the Legbesgue measure. Thus our results show the precise limitations imposed on the ergodic properties of mapping on $\mathbb{T}$ by smoothness conditions.

1 Basic facts about orbit-equivalence

A non-singular isomorphism of one Lebesgue measure space $(X, \mathcal{B}, \mu)$ onto another, $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$, is an (almost everywhere defined) invertible, bimeasurable map $\psi : X \mapsto \tilde{X}$ which carries μ onto a measure equivalent to $\tilde{\mu}$. An isomorphism of $(X, \mathcal{B}, \mu)$ onto itself is called a non-singular automorphism or a non-singular transformation. A non-singular system is a quadruplet $(X, \mathcal{B}, \mu, \varphi)$ where $(X, \mathcal{B}, \mu)$ is a Lebesgue measure space and φ a non-singular automorphism on it.

A non-singular system is ergodic if there exist no non-trivial φ-invariant measurable sets.

Unless specified otherwise *all the systems we deal with are assumed to be non-singular and ergodic* and we shall usually omit these adjectives.

Definition 1.1. The systems $(X, \mathcal{B}, \mu, \varphi)$ and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{\varphi})$ are *isomorphic* if there exists an isomorphism ψ of $(X, \mathcal{B}, \mu)$ onto $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ such that

(1.1) $$\psi\varphi x = \tilde{\varphi}\psi x \qquad \text{a.e.}$$

The notion of orbit-equivalence, also referred to as “weak-equivalence" or “Dye-equivalence" is much coarser:

Definition 1.2. The systems $(X, \mathcal{B}, \mu, \varphi)$ and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{\varphi})$ are *orbit-equivalent* if there is an isomorphism ψ of $(X, \mathcal{B}, \mu)$ onto $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ such that

(1.2) $$\psi(\{\varphi^j\}_{j\in\mathbb{Z}}) = \{\tilde{\varphi}^j\psi x\}_{j\in\mathbb{Z}} \qquad \text{a.e.}$$

Notice that (1.2) is equality of sets and the requirement is that (almost all) φ-orbits be mapped by ψ onto $\tilde{\varphi}$-orbits without regard to the order of points along an orbit.

Definition 1.3. A system $(X, \mathcal{B}, \mu, \varphi)$ is of

(a) *type II*$_1$ if there exists a φ-invariant *probability* measure on (X, B) equivalent to μ;

(b) *type II*$_\infty$ if there exists a φ-invariant, *infinite* measure on (X, B) equivalent to μ;

(c) *type III* if it is not of type II (that is: $\mathrm{II}_1 \cup \mathrm{II}_\infty$).

It is easy to see that these types are preserved under orbit equivalence; types II_1 and II_∞ are in fact complete invariants.

Theorem 1.4 (Dye). *Any two systems of type II*$_1$ *are orbit-equivalent; any two systems of type II*$_\infty$ *are orbit-equivalent.*

Type III is far from being a complete invariant. It can be further subdivided into types III_λ, $0 \le \lambda \le 1$, by means of the so-called ratio-set, as was done by Krieger who also showed that for $\lambda \ne 0$, III_λ is again a complete orbit-equivalence invariant while III_0 is far from it.

Induced systems. Let $(X, \mathcal{B}, \mu, \varphi)$ be a systme and $A \in B$ a set of positive μ-measure. We denote by B_A the algebra of B-measurable subsets of A; by μ_A the restriction of μ to B_A and by φ_A the mapping induced by φ on A, that is, $\varphi_A(x_= \varphi^{N(x)}(x)$ for $x \in A$ where $n(x)$ is the smallest positive integer n for which $\varphi^n(x) \in A$. We refer to $(A, B_A, \mu_A, \varphi_A)$ as the system induced on A by $(X, \mathcal{B}, \mu, \varphi)$.

Theorem 1.5. *If* $(X, \mathcal{B}, \mu, \varphi)$ *is of type II*$_1$ *or III, then* $(A, \mathcal{B}_A, \mu_A, \varphi_A)$ *is orbit-equivalent to* $(X, \mathcal{B}, \mu, \varphi)$.

Remark. If $(X, \mathcal{B}, \mu, \varphi)$ is of type II_∞ then $(A, \mathcal{B}_A, \mu_A, \varphi_A)$ is either II_∞ or II_1.

Odometers. Odometers, also know as Adding-machines, serve as prototypes for orbit-equivalence classes (see Theorem 1.6 below). Let $\{d_k\}_{k=1}^{\infty}$ be a sequence of positive integers and denote by $\mathcal{O}(\{d_k\})$ the (topological) system (X, B, φ) where X is the compact metrisable space $\Pi_{k=1}^{\infty}(0, 1, \cdots, d_k - 1)$, B is the Borel algebra on X and φ is "addition of 1 with carry-over", that is, if $x \in X, x = \{x_k\}_{k=1}^{\infty}$ with $0 \leq x_k < d_k$ for all k, write $r(x) = \inf\{k; x_k < d_k - 1\}$, the $\varphi(x) = \{y_k\}$ where $y_k = 0$ for $k < r(x)$, $Y_k = x_k + 1$ for $k = r(x)$ and $y_k = x_k$ for $k > r(x)$. (Thus, in particular $\varphi(\{d_k - 1\})$ is the zero sequence.) Let μ be a continuous measure on (X, B) which is ergodic and quasi-invariant under φ. We refer to $\big(X, \mathcal{B}, \mu, \varphi\big)$ as a (measured) odometer and denote it by $\mathcal{O}(\{d_k\}, \mu)$.

Theorem 1.6. *Every (ergodic, non-singular) system is orbit-equivalent to some measured odometer.*

Let $\mathcal{O}(\{d_k\})$ be an odometer and assume that v_k is a probability measure on $(0, 1, \cdots, d_k - 1)$ such that the probability of every digit is positive and the product measure $v = \Pi v_k$ is non-atomic on $\mathcal{O}\{d_k\})$. It is easy to check that v is automatically ergodic and quasi-invariant under φ. We refer to $\mathcal{O}(\{d_k\}, v)$ as *odometer of product type* and denote it also by $\mathcal{O}(\{d_k\}, \{v_k\})$.

Definition 1.7. A system is of *product type* if it is orbit-equivalent to some $\mathcal{O}(\{d_k\}, \{v_k\})$.

It was shown by Krieger, with further discussion and examples by Connes and Woods, that there exist systems which are *not* of product type.

Formally different odometers may well be orbit-equivalent; e. g., all measure preserving odometers. The following fact will be esential to our contruction.

Theorem 1.8. *Let $\mathcal{O}(\{d_k\}, \{v_k\})$ be an odometer of product type. Let N_k be positive integers and let v_k^* be a probability measure on $A = \{0, 1, \cdots, N_k d_k - 1\})$ such that there exists a partition $A = \bigcup A_j$ where $A_j^{\#} = N_k$ and $v_k^*(n) = N_k^{-1} v(j)$ for $n \in A_j$, $j = 0, \cdots, d_k - 1$. Then $\mathcal{O}(\{N_k d_k\}, \{v_k^*\})$ is orbit-equivalent to $\mathcal{O}(\{d_k\}, \{v_k\})$.*

Another way to state Theorem 1.8 is: if σ_k is the equidistributed probability on $\{0, \cdots, N_k - 1\}$ and if we put $m_{2k-1} = d_k$, $m_{2k} = N_k$, $\mu_{2k-1} = v_k, \mu_{2k} = \sigma_k$, then $\mathcal{O}(\{m_k\{, \{\mu_k\})$ is orbit-equivalent to $\mathcal{O}(\{d_k\}, \{v_k\})$.

2 Diffeomorphisms of the circle

We refer to the expository part of [2] for most of the background material on diffeomorphisms of the cirlce which we use here.

Lef f be an orientation preserving diffeomorphism of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $\alpha = \rho(f)$ the rotation number, assume that α is irrational, and write $[a_1, a_2, \cdots]$ for the continued fraction expansion of α (that is

(2.1) $$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

In the case that $\{a_n\}$ is bounded and $f \in C^3$, f is conjugate to the rotation by α via a diffeomeorphism and the unique f-invariant measure on $\mathbb{T}$ is equivalent to the Lebesgue measure μ; thus $(\mathbb{T}, b, \mu, f)$ is of type II_1. This is a special case of Herman's theorem [1]. We shall therefore assume from now on that $\{a_n\}$ is unbounded. Under this and the assumption $f \in C^2$ we shall prove that $\mathbb{T}, Bu, \mu, f)$ is orbit-equivalent to an odometer of product type and that, conversely, any such odometer is equivalent to some $f \in C^\infty$.

As usual, we put $p_n/q_n = [a_1, \cdots, a_n]$, the convergents of α. It is well known (cf. [2], I. 4) that one can get p_n and q_n directly from the orbit, under the rotation R_α, of a point, say $t = 0$, on $\mathbb{T}$. In particular, the sequence $\{q_n\}$ of the denominators is simply the sequence of "closest return", that is, those integers k such that $(k \mod 1)$ is closer to zero that nay $(j\alpha \mod 1)$ with $0 < j < k$. We denote by d_n the distance of $q_n\alpha \mod 1$ to zero and recall that $a_n = [d_{n-2}/d_{n-1}]$ and $q_n = a_nq_{n-1} + q_{n-2}$. If we take an interval I of length d_n, say $[0, q_n\alpha)$ if $n = 2m$ or $[q + n\alpha, 0)$ if $n = 2m + 1$, and consider its images $R^j_\alpha(I), - \leq j < q_{n+1}$, we see that these are disjoint (no two points in $\{j_\alpha\}, j = 0, \cdots, q_{n+1} - 1$ are closer than d_n). Also, their union covers almost the entire circle; only q_n intervals of length d_{n+1} are left uncovered, each one contained in some $R^j_\alpha(I)$ with $q_{n+1} \leq j < q_{n+1} + q_n$. Everything except sizes is carried over by conjugation; thus if $I = [t_o, f^{a_n}(t_0))$, then $\{f^j(I)\}_{j=0}^{q_{n+1}-1}$ are disjing and $\{f^j(I)\}_{j=0}^{q_{n+1}+q_n-1}$ covers $\mathbb{T}$.

The following observation is due to Finzi and, later, Herman:

Lemma 2.1. *Denote* $V = \mathit{var}(\log Df)$. *Assume* $t, \tau \in [t_0, f^{q_n}(t_0))$. *Then for* $0 \leq j < q_{n+1}$

$$e^{-V} \leq \frac{Df^j(t)}{Df^j(\tau)} \leq e^V. \tag{2.2}$$

Here, and later, Df^j denotes the derivative (of the jth iterate of f).

Proof. By the chain rule $\log Df^j(t) = \sum_{k=0}^{j-1} \log Df \circ f^k(t)$ and similarly for $\log Df^j(\tau)$, hence

$$\log Df^j(t) - \log Df^j(\tau) = \sum_{k=0}^{j-1} (\log Df \circ f^k(t) - \log Df \circ f^k(\tau)) \tag{2.3}$$

and since $\{f^k(t), f^k(\tau)\}_{k=0}^{j-1}$ are disjoint, we obtain

$$|\log Df^j(t) - \log Df^j(\tau)| \leq V \tag{2.4}$$

which is equivalent to (2.1).

Katok used a variant of this lemma to prove that, if $f \in C^2$ and $\rho(f)$ is irrational, $(\mathbb{T}, B, \mu, f)$ is ergodic. In fact, if $E \subset \mathbb{T}$ is f-invariant and of positive Lebesgue measure, take a density point t_0 of E and apply the lemma for large values of n. The relative measure of $\mathbb{T} \setminus E$ in $f^j([t_0, f^{q_n}(t_0)))$ is no more than e^V times its relative measure in $[t_0, f^{q_n}(t_0))$ which is arbitrarily small. This is true for $0 \leq j <$

q_{n+1} which, as we pointed out above, does not quite cover $\mathbb{T}$. However, if we continue with $f^j\big([f^{q_n}(t_0), f^{2q_n}(t_0))\big), 0 \le j < q_n$, we obtain that the measure $\mathbb{T} \setminus E$ is arbitrarily small and $E = \mathbb{T}$ (mod 0).

We shall use the lemma to show that (when $\{a_n\}$ is unbounded f induces on subsets of $\mathbb{T}$ of measure arbitrarily close to 1 an odometer of product type. Let us review the case $f = R_\alpha$. For even n denote

$$E_n = \bigcup_{J=0}^{q_{n+1}-1} R_\alpha^j([0, q_n\alpha)). \tag{2.5}$$

We have already mentioned above that the union defining E_n is disjoint and that $\mathbb{T} \setminus E_n$ is the union of q_n intervals of length d_{n+1}. Since $q_n d_{n-1}$ is the measure of E_{n-1}, it is less than one, and $q_n d_{n+1} \sim (a_{n+1}a_{n+2})^{-1}$; it follows that if $a_{n+1}a_{n+2}$ is large, the measure of E_n is close to one. If $\{a_n\}$ is unbounded, we can take a sequence $\{n_j\{$ of even integers such that $a_{n_j+1} \cdot a_{n_j+2} > 2^j$ and $E \cup_{j=10}^{\infty} E_{n_j}$ has positive measure. We claim that R_α induces on E and odometer. If we denote by Q_j the partition of E by the “components" of E_{n_j} , it is clear that $V_{j=10}^{\infty} Q_j$ is the Borel field of E and, on the other hand, $R_\alpha | E$ is a cyclic permutation of the elements of Q_j. This means tht we have an odometer as claimed.

Returning to $f \in C^2$ with $\rho(f) = \alpha$, we obtain the odometer there by conjugation. We could have nothing more to say were we interested in the action of f on the invariant measure, which is the conjugation image of dt. Since we are studying the action of f on the Lebesgue measure itself, we start again from the beginning.

We need to quote two results:

Lemma 2.2 (Denjoy). *If $f \in C^2, \rho(f) = \alpha, p/q$ is a reduced rational fraction satisfying $|\alpha - p/q| < 1/q^2$ and $V = var(\log Df)$, then*

$$e^{-V} \le Df^q \le e^V. \tag{2.6}$$

Proof. See [2] theorem 3.4.

Lemma 2.3 (Herman). *If $f \in C^2$, there exists a sequence $\{h_n\}, h_n \in C^2$ such that writing $f_n = h_n \circ f \circ h_n^{-1}$ we have $V_n = var(\log Df_n) \to 0$.*

Proof. See [1]; the sequence $\{h_n\}$ is given explicitly by $\tilde{h}_n = (1/n)\sum_{j=0}^{n-1}(\tilde{f}^j - j\alpha)$ where $\tilde{f}$ is the lifting of f to $\mathbb{R}$, and $h_n = \tilde{h}_n \mathrm{mod} 1$.

Corollary 2.4. *If $f \in C^2$, then $Df^{q_n} \to 1$ uniformly.*

Proof. We use also the fact that f is conjugate to R_α which means that $f^{q_n}(t) \to t$ uniformly as $n \to \infty$. Now take $0 < \epsilon < 1$ and m large enough so that (Lemma 2.3) var$(\log Df_m) < \epsilon/10$. By Lemma 2.2

$$|Df_m^{q_n} - 1| < \frac{\epsilon}{2}.$$

Since $f^{q_n} = h_m^{-1} \cdot f_m^{q_n} \cdot h_m$ we have

$$Df^{q_n}(t) = (Dh_m^{-1} \cdot f_m^{q_n} \cdot h_m(t)) \cdot (Df_m^{q_n} \cdot h_m(t)) \cdot Dh_m(t)$$

and for large values of $n, f_m^{q_n} \cdot h_m \sim h_m$ so that the two extreme factors just about cancel each other while the center factor is within $\epsilon/2$ of one. *QED*

Lemma 2.5. *Let g be a diffeomorphism of $\mathbb{T}$. Assume $|Dg - 1| < \epsilon$ and let $B \subset \mathbb{T}$ be such that $g^{-j}(B)$ are disjoint for $j = 0, 1, \cdots, M$. Then $\mu(B) < \epsilon/(1-(1-\epsilon)^{M+1})$.*

Proof. $\mu(g^j(B)) \geq (1-\epsilon)^j \mu(B)$ and $\sum_0^M \mu(g^j(B)) \leq 1$.

We are now ready to prove

Theorem 2.6. *Assume $f \in C^2$ and $\alpha = \rho(f)$ has unbounded continued fraction coefficients. Then f induces an odometer of product type on subsets of positive Lebesgue measure on $\mathbb{T}$.*

REMARK: By Theorem 1.5 this means that (X, B, μ, f) is itself of product type.

Proof. Put $\tilde{E}_n = \cup_{j=0}^{q_{n+1}^{-1}} f^j([0, f^{a_n}(0)))$ and notice that for ever $n, f^{q_n}(0)$ is just "right" of zero and $\tilde{E}_n$ is a disjoint union. We have

$$\tilde{F}_n = \mathbb{T} \setminus \tilde{E}_n = \cup_{j=0}^{q_n - 1} f^j([f^{q_{n+1}}(0), 0))$$

and since $\{f^j([f^{q_{n+1}}(0), 0))\}$ is disjoint for $0 \leq j < q_{n+2}$, we have $\{f^{lq_n})(\tilde{F}_n)\}$ disjoint for $l < q_{n+2}q_n^{-1}$. By Corollary 2.4 and Lemma 2.5 (applied to $g = f^{q_n}$ and $B = F_n$) we have $\mu(\tilde{F}_n) \to 0$ as $n \to \infty$ on a subsequence such that $q_{n+2}q_n^{-1} \to \infty$. We can therefore choose a sequence $\{n_k\}$ of even integers such that

$$\sum \mu(\tilde{F}_{n_k} < \frac{1}{10}. \tag{2.7}$$

In fact, we impose on $\{n_k\}$ still another condition. By Lemma 2.3 there exists a sequence of C^2 diffeomorphisms of $\mathbb{T}$, $\varphi_k(= h_{m_k}$ in the notation of Lemma 2.3) such that writing $f_k = \varphi_k \cdot f \cdot \varphi_k^{-1}$ we have var$(\log Df_k) < 2^{-k}$. With no loss of generality we may normalized φ_k by $\varphi_k(0) = 0$. As the maximal length of the components of $\tilde{E}_n$ tends to zero when $n \to \infty$, we can take $\{n_k\}$ such that, besides (2.2), the following is true:

$$\max D\varphi_k(t)/minD\varphi_k(t) < 1 + 2^{-k}, \tag{2.8}$$

the maximum and the minimum taken for t in any (same) component of $\tilde{E}_{n_k}$.

We now take $\tilde{E} = \cap E_{n_k}$, write $I_k = \tilde{E} \cap [0, f^{q_{n_k}}(0))$, $\tilde{Q}_k$ denotes the partition of $\tilde{E}$ an odometer for which $\{Q_k\}$ is the sequence of basic partitions, that is, Q_k, is the partition according to the first k digits. For every k, I_k is the "base", that is, the set corresponding to the condition that the first k digits all vanish. The sequence $\{f^j(I_k)\}$ ocveres the space as $j = 0, \cdots, q_{n_k+1} - 1$ and a measure v on $\tilde{E}$ is a product measure on the odometer if for every k and $0 \leq j < q_{n_k+1}$ the Radon-Nikodym derivative $df^{-j}v/dv$ is constant on I_k. We shall show now that for the Lebesgue measure μ the derivative condition is almost satisfied and thus will be able to construct $v \sim \mu$ for which it is true.

Assume $t, \tau \in I_i, 0 \leq j < q_{n_k+1}$ and write $f = \varphi_k^{-1} \cdot f_k \cdot \varphi$: we have

$$Df^j(t) = D\varphi_k^{-1}(f_k^j \circ \varphi_k(t)) \cdot Df_k^j(\varphi_k(t)) \cdot D\varphi_k(t).$$

$$Df^j(\tau) = D\varphi_k^{-1}(f_k^j \circ \varphi_k(\tau)) \cdot Df_k^j(\varphi_k(\tau)) \cdot D\varphi_k(\tau).$$

We estimate the ratios $D\varphi_k(t)/D\varphi_k(\tau)$ and $D\varphi_k^{-1}(f_k^j \circ \varphi)k(t))/D\varphi_k^{-1}(f_k^j \circ \varphi_k(\tau))$ by (2.3); use Lemma 2.1 to estimate $Df_k^j(\varphi_k(t))/Df_k^j(\varphi_k(\tau))$, noticing that, by conjugation, $\varphi_k(t)$ and $\varphi_k(\tau)$ both belong to $[0, f_k^{q_{n_k}}(0))$, and we obtain

$$Df^j(t)/df^j(\tau) \leq (1 + 2^{-k})^2 e^{2-k} = 1 + \epsilon_k. \tag{2.9}$$

Take $k = 1$ and consider the measure μ_1 which agrees with μ on I_1 while on $f^j(I_1)$, $1 \leq j < q_{n_1+1}, \mu_1 = c_j f^j \mu$ where $c_j = \mu(f^j(I_1))/\mu(I_1)$ (so that $\mu_1(f^j(I_1)) = \mu(f^j(I_1)))$. It is clear that $(df^{-j}\mu_1)/d\mu_1 = c_j$ on I_1 and by (2.4) $\mu_1 = g_1\mu$ with $(1 + \epsilon_1)^{-1} \leq g_1(t) \leq 1 + \epsilon_1$.

We now define μ_2: μ_2 agrees with μ on I_2 while on any $f^j(I_2)$ which is contained in $I)1$, $\mu_2 = (\mu(f^j(I_2))/\mu(I_2))f^j\mu$. Once μ_2 is defined on I_1, we extend it to $\tilde{E}$ by the rule that μ_2 on $f^j(I_1)$ is $c_j f^j \mu_)2$, with the same constants c_j used in the definition of μ_1. Again by (2.4) we have $\mu_2 = g_2\mu$ on I_1 with $(1 + \epsilon_2)^{-1} \leq g(t) \leq 1 + \epsilon_2$ and extending the definition of g_2 to $\tilde{E}$ by $g_2(t) = g_2 \circ f^{-1}$ on $f^j(I_1)$, we obtain $\mu_2 = g_2\mu_1 = g_2 g_1 \mu$. Now for μ_2 the Radon-Nikodym derivatives $(df^{-j}(\mu_2))/d\mu_2$ are constant on I_1 for $j < q_{n_l+1}$ and on I_2 for $j < q_{n_2+1}$. Continuing in the same manner, we obtain functions g_k on $\tilde{E}$ such that $(1 + \epsilon_k)^{-1} \leq g(t) \leq 1 + \epsilon_k$ and writing $\mu_k = \pi_{j=1}^k g_j \mu$ the condition on the Radon-Nikodym derivatives is satisfied on $I_1, \cdots, I_k$. Writing $v = \lim_{k\to\infty} \mu_k$, it is clear that v is the product measure on $\tilde{E}$ and

$$\prod^{\infty}(1 + \epsilon_k)^{-1} \leq \frac{dv}{d\mu} \leq \prod^{\infty}(1 + \epsilon_k).$$

QED

We now turn to show that any odometer of product type is orbit-equivalent to some C^∞ diffeomorphism of the circle. The construcitn we use is that of [2] with minor refinement; but before we start, let us make it very clear what it is we are trying to construct. The odometer $\mathcal{O}$ is given by a sequence $\{d_k\}_{k=1}^\infty$ of positive integers and a sequence $\{v_k\}$ where v_k is a probability measure on $\{0, 1, \cdots, d_k - 1\}$ such that the probability of every digit is positive and $\prod v)k$ is a non- atomic measure. Staying withing the same orbit-equivalence class, we are allowed, by Theorem 1.8, to replace $\mathcal{O}(\{d_k\}, \{v_k\})$ by $\mathcal{O}(\{N_k d_k\}, \{v_k^*\})$ where $\{N_k\}$ is an arbitrary sequence of positive integers and v_k^* is a probability measure on $\{0, \cdots, N_k d_k - 1\}$ obtained from v_k by divind each point mass of v_k into N_k equal masses. Becuase of this and of Theorem 1.5, it is enough, when we want to show that some $f \in C^\infty$ is orbit-equivalent to $\mathcal{O}(\{d_k\}, \{v_k\})$, to show that f induces on some set of positive measure an odometer isomorphic to $\mathcal{O}(\{N_k d_k\}, \{v_k^*\{)$.

Theorem 2.7. *For every odometer of product type $\mathcal{O} = \mathcal{O}(\{d_k\}, \{v_k\})$ the set of C^∞-diffeomorphisms of $\mathbb{T}$ which are orbit- equivalent to $\mathcal{O}$ is dense (C^∞) in the set of all C^∞-diffeomorphisms with irrational rotation number on $\mathbb{T}$.*

Proof. As we mentioned before, the proof is just a minor refinement of that of theorems 3.2 and 3.3 in [2]. We shall confine ourselves to a description of the construction and leave the details to the reader who can fill them in just as in §3 of [2].

For a diffeomorphism g with $\rho(g) = \alpha$ we denote by $P_n(g)$ the partition of $\mathbb{T}$ by $\{g^j(0)\},\ j = 0, 1, \cdots, q_n - 1$ (q_n being the nth denominator of $\alpha = \rho(g)$). What we are trying to achieve is : given the data $(\{d_k\}, \{v_k\})$ of the odometer, there exists a sequence of integers $\{n_k\}$ such that for $n_k < n < n_{k+1}$ each interval of $P_{n_k}(f)$ is divided in $P_n(f)$ into essentiall equal intervals, while each interval of $P^{-1}_{n_k-1}(f)$ is divided in $P_{n_k+1}(f)$ into $M_{k+1}d_{k+1}$ "good" intervals plus some additional intervals. The union of the "good" intervals fills up at least $(1 - k^{-1})$ of the measure of the $P_{n_{k+1}-1}(f)$ interval containing them, and can be divided into d_{k+1} groups of M_{k+1} invervals each, such that all the intervals in the jth group are essentially equal and have relative length equal to $v_{k+1}(j), j = -, \cdots, d_{k+1} - 1$. If we denote by E_k the union of all the "good" subintervals of $P_{n_k+1}(f)$ which are contained in E_{k-1}, then $\mu(E_k) > (1 - k^{-2})\mu(E_{k-1})$, and setting $E = \cap_k E_k$, we obtain that $\mu(E) > 0$ and that f induces on (E, μ) and odometer isomorphic to $\mathcal{O}(\{N_k d_k\}, \{v_k^*\})$ where $N_k = M_k$ times the number of $P_{n_{k+1}-1}$ intervals in each P_{n_k} interval.

Now to the construction. We start with any $f_0 \in C^\infty$ such that $\rho(f_0)$ is irrational and approximate it, arbitrary well, by $\bar{f}_0 = R_\epsilon \circ f_0$ such that $\rho(\bar{f}_0)$ is irrational and has bounded continued fraciton coefficients. By Herman's theorem (quoted as theorem 3.1 in [2] for analytic funcitons but valied equally for C^∞) there exists a C^∞ close to $R_{\rho(\bar{f}_0)}$, xxxx which induces an odometer equivalent to $(\{d_k\}), \{v_k\})$, then $h_0^{-1} \circ g \circ h_0$ will be close to $\bar{f}_0$, hence to f_0, and will induce the same odometer. Thus we can limit a sequnce f_k such that $\alpha_k = \rho(f_k)$ is irrational and its continued fraction picture described above is valid for f_k up to n_k. Assume that we have constructed f_k, $h_k \in C^\infty$ such that $f_k = h_k^{-1} \circ R_{\alpha_k} \circ h_k$ is very close to f_k in C^∞. Given $\epsilon = \epsilon_k$, we now choose $n - n_{k+1} - 1$ large even integer so that

(a) $C^\infty\text{-dist}(f_k^*, f_k) < \epsilon_k$,

(b) Dh_k is essentially constant on every interval of $P_n(f_k^*)$,

(c) to be explained later.

We shall choose $a_k^*(n+1)$ very large, and by the given conjugation and (b), every $P_n(f_k^*)$ interval is divided in $P_{n_1}(f_k^*)$ into $a_k^*(n+1)$ (or one more) essentially equal intervals.

Let $I \in P_n(f_k^*), |I| > q_n^{-1}$ and divide it into d_{k+1} intervals $I_0, \cdots, I_{d_{k+1}^{-1}}$ separated by intervals $J_0, \cdots, J_{d_{k+1}}$, so that I_l lies between J_l and J_{l+1}, such that $|I_l| = (1 - k^{-3})v_{k+1}(l)|I|$ and $|J_l| = (d_{k+1} + 2)^{-1}k^{-3}|I|$.

We now consider C^∞ functions φ carried out by I which are constant on each I_l and monoton on each J_l and also have very small C^∞ distance from zero. For a proper choice of $A)k^*(n+1)$ and such φ we shall put $f_{k+1} = f_k^* + \varphi$. For any such choice, the orbit of 0 under f_{k+1} is the same as that under f_k^* as long as it does not enter I so that $P_n(f_{k+1}) = P_n(f_n^*)$. On I we have $Df_{k+1}^{q_n} \simeq D(f_k^*)^{q_n} + D\varphi$, and since the $P_{n+1})f_{k+1})$ intervals in I are mapped onto each other by powers of $f_{k+1}^{q_n}$, we see that any two such intervals within the same I_l are essentailly equal (since $D\varphi = 0$ in I_l). However, when we move from I_l to I_{l+1}, the orbit under $f_{k+1}^{q_n}$ crosses J_{l+1}, and if $D\varphi \neq 0$ there, the ratio of the lengths of a $P_{n+1}(f_{k+1})$ interval in I_{l+1} and one in I_l will be $(1 + (D\varphi)_l)^{m_l}$ where $(\bar{D\varphi})_l$ is some sort of mean value of $d\varphi$ on J_{l+1} and m_l is the number of times an orbit under $f_{k+1}^{q_n}$ falls in J_{l+1} between I_l and I_{l+1}. If $a_k^*(n+1)$ is very large so that we move by very small steps, we can obtain by adjustment of φ, even though $(\bar{D\varphi})_l$ are very small, any preassigned ratio between $P_{n_1}(f_{k+1})$ intervals in I_{l+1} and those in I_l. We adjust it so that the ratio is $v_{k+1}(l+1)/v_{k+1}(l)$, so that each I_l is divided in $P_{n+1}(f_{k+1})$ into the same number, say M_{k+1}, of equal intervals.

We still have some freedom in adjusting the rotation number α_{k+1} of f_{k+1}. It is clear that $a_{k+1}(j) = a_k^*(j) = a_k(j)$ for $j \leq n$. It is also clear that $a_{k+1}(n+1)$ is very learge and that we can change it by one or two via small modification of φ or by replacing f_{k+1} by $R_\eta \circ f_{k+1}$ without modifying the description of $P_{n+1}(f_{k+1})$ beyond an alloweable error. By continuity of the rotation number we may impose therefore that $a_{k+1}(j) = 100$ for $j > n+1$.

We still have to describe conditions (c) in the choice of $n = n_{k+1}$, and we need it in order to show that each interval of $P_{n_1}(f_{k_1})$ is divided in $P_r(f_{k+1})$ for any $r > n_{k+1} + 1$, as it would be for $R_{\alpha_{k+1}}$. The proof is that of theorem 3.3 in [2], the key to which is lemma 3.4 there which states that if $n = n_{k+1}$ is large enough (and this is our condition (c)), then

$$||df_{k+1}^{q_m} - 1||_C \leq 3^{-m} \qquad \text{for } m > n_{k+1}.$$

If I' is any other interval of $P_n(f_{k+1})$ on I' from that on I by transporting the latter by f_{k+1}^j for some $|j| < q_n$. The fact that $f_{k+1} = h_k^{-1} \circ R_{\alpha_k^*} \circ h_k + \varphi$ and that φ was chosen arbitrarily small after q_n was known allows us to assume $Df_{k+1}^j \approx d(f_k^*)^j$ and since by our choice Dh_k was essentially constant on I and I', we have Df_{k+1}^j essentially constant of I and the picture on I' is similar to that on I. This concludes the proof. *QED*

References

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