Math 205C Homework assignment #7

due on May 27, 1998.

7.1 Let $f, g \in L^2(\mathbb{R})$ and assume that $\hat{f}(\xi) = 0$ implies $\hat{g}(\xi) = 0$ for almost all $\xi \in \hat{\mathbb{R}}$.

Prove that g can be approximated on $L^2(\mathbb{R})$ by linear combinations of translates of f.

7.2 For measurable sets $E \subset \hat{\mathbb{R}}$ denote: $H_E = \{f \in L^2(\mathbb{R}) : \hat{f} = \mathbb{1}_E \hat{f}\}$.

Prove that H_E is a closed translation-invariant subspace of $L^2(\mathbb{R})$, and that every closed translation-invariant subspace of $L^2(\mathbb{R})$ is obtained this way.

7.3 Let $S = [0,1] \times [0,1]$ and $L^2(S)$ the L^2 space for the product measure $dx\,dy$.

Assume $K(x,y) \in L^2(S)$.

a. Show that the integral operator $\mathbf{k}$ defined on $L^2([0,1])$ by

$$\mathbf{k}f(x) = \int K(x,y) f(y) dy$$

is well defined and bounded, its norm bounded by $\|K\|_{L^2}$.

b. Prove that $\mathbf{k}$ is compact.

Hint: Show that the functions of the form $f \otimes g = f(x)g(y)$ span $L^2(S)$.

c. Let $K(x,y)$ be bounded and measurable in $0 \le y < x \le 1$, and denote by $\mathbf{v}$ the Volterra integral operator

$$\mathbf{v}f(x) = \int_0^x K(x,y) f(y)\,dy.$$

Prove that the spectrum of $\mathbf{v}$ is $\{0\}$.

7.4 Assume that $B \subset C(\mathbb{T})$ is closed in $L^2(\mathbb{T})$. Prove that it is finite dimensional.

Let K be such that $\|f\|_\infty \le K\|f\|_2$ for all $f \in B$.

Let $\{\varphi_n\}_1^N$ be orthonormal in B and observe that

$$\max_{\sum a_j^2 \le 1} \left|\sum a_j \varphi_j(x)\right| \le K \tag{7.1}$$

i.e. $\sum |\varphi_j|^2 \le K^2$, integrate and $N \le K^2$.

7.5 The Hardy–Littlewood maximal function of an integrable function f on $\mathbb{R}$ (or on some interval $I \subset \mathbb{R}$) is defined by

$$M_f(x) = \sup_{\substack{h \\ (x-h,x+h)\subset I}} \frac{1}{2h} \int_{\lambda-h}^{x+h} |f(t)|dt.$$

a. Prove that M_f is of weak-L^1-type, that is

$$\mu\left(\{x; M_f(x) > \lambda\}\right) \leq \frac{c}{\lambda}.$$

b. For $p > 1$, if $f \in L^p(I)$ then M_f is of weak-L^p-type, that is

$$\mu\left(\{x; M_f(x) > \lambda\}\right) \leq \frac{c}{\lambda^p}.$$