## Math 205C    Homework assignment #5

due on May 6, 2009.

**5.1** Let $B$ be homogeneous on $\mathbb{T}$ and $B^*$ its dual.

**a.** Show that $S \sim \sum a_n e^{int}$ is the Fourier series of some $\mu \in B^*$ if, and only if, $\|\sigma_N(S)\|_{B^*}$ is bounded as $N \to \infty$.

**b.** Show that a trigonometric series $\sum a_n e^{int}$ such that $\sum_{-N}^{N} a_n e^{int} \geq 0$ for all $N$ and $t \in \mathbb{T}$ is a Fourier–Stieltjes series of a positive measure.

**c.** Denote $\mathsf{K}_{n,\tau}(t) = \mathsf{K}_n(t-\tau)$. Show that for every $\mu \in M(\mathbb{T})$

$$\sigma_n(\mu,\tau) = \overline{\langle \mathsf{K}_{n,\tau}, \mu \rangle}\,.$$

Deduce that $\sigma_n(\mu,\tau) \geq 0$ if $\mu$ is positive.

**5.2** **a.** Show that a measure $\mu \in M(\mathbb{T})$ is absolutely continuous if and only if it *translates continuously*, i.e., $\lim_{\tau\to 0}\|\mu_\tau - \mu\|_{M(\mathbb{T})} = 0$, where $\mu_\tau$, defined by $\mu_\tau(E) = \mu(E-\tau)$, is the translate of $\mu$ by $\tau$.

**b.** Let $\mu \in M(\mathbb{T})$, and $\alpha$ an irrational multiple of $\pi$. Prove that if $\mu_\alpha - \mu$ is continuous, then $\mu$ is continuous, and if $\mu_\alpha - \mu$ is absolutely continuous, then $\mu$ is absolutely continuous.

**5.3** **a.** Show that for all $n$ and $t$, $\left|\sum_1^n j^{-1}\sin jt\right| \leq \frac{1}{2}\pi + 1$.

Hint: Consider $f(t) = t/2$ in $[0, 2\pi)$.

**b.** Let $\alpha, \beta \in \mathbb{T}$, let $N$ be an integer, and $\mu$ the measure carried by the arithmetic progression $\{\alpha + j\beta\}_{j=-N}^{N}$, which places the mass zero at $\alpha$ and the mass $j^{-1}$ at $\alpha + j\beta$, for $1 \leq |j| \leq N$. Show that $\|\mu\|_{PM(\mathbb{T})} \leq \pi + 2$.

**c.** Let $f \in A(\mathbb{T})$ be real valued and monotone in a neighborhood of $t_0 \in \mathbb{T}$. Show that $|f(t) - f(t_0)| = O\big((\log|t-t_0|^{-1})^{-1}\big)$ as $t \to t_0$.

**5.4** **a.** Let $\mu$, $\mu_n \in M(\mathbb{T})$, $n = 1, 2, \ldots$. Prove that $\mu_n \to \mu$ in the weak-star topology if, and only if, $\|\mu_n\|_{M(\mathbb{T})} = O(1)$ and $\hat{\mu}_n(j) \to \hat{\mu}(j)$ for all $j$.

**b.** By definition, a sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathbb{T}$ is *equidistributed* or *uniformly distributed* if for any arc $I \subset \mathbb{T}$ we have $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} \mathbb{1}_I(\xi_n) = (2\pi)^{-1}|I|$. Prove the following statements:

***b.1.*** *Weyl's criterion.* $\{\xi_n\}_{n=1}^{\infty} \subset \mathbb{T}$ is uniformly distributed if and only if the measures $\mu_n = n^{-1}\sum_1^n \delta_{\xi_j}$ converge in the weak-star topology to $(2\pi)^{-1}dt$, i.e., if $n^{-1}\sum_1^n e^{il\xi_j} \to 0$ for all integers $l \neq 0$.

***b.2.*** If $\alpha$ is an irrational multiple of $\pi$, the sequence $\{n\alpha\}$ is equidistributed on $\mathbb{T}$.

**5.5** ***(van der Corput)*** Given a bounded sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers.

**a.** Prove that for $N \in \mathbb{N}$ and $1 < H < N$,

$$\Big|\frac{1}{N}\sum_{n=1}^{N} a_n\Big|^2 \leq \frac{1}{N}\sum_1^N \Big|\frac{1}{H}\sum_{j=1}^{H} a_{n+j}\Big|^2 + O\left(\frac{H}{N}\right). \tag{5.1}$$

**b.** The terms that occur in $\left|\sum_{j=1}^{H} a_{n+j}\right|^2$ have the form $a_{n+j}\overline{a_{n+k}}$ with both $j$ and $k$ in $[1,H]$.

Check: the product $a_l\bar{a}_m$ appears for $\max(H - |l-m|, 0)$ values of $n$, hence

$$\Big|\frac{1}{N}\sum_{n=1}^{N} a_n\Big|^2 \leq \frac{1}{H}\sum_{k=-H}^{H}\left(1 - \frac{|k|}{H}\right)\frac{1}{N}\sum_{n=1}^{N} a_n\overline{a_{n+k}} + O\left(\frac{H}{N}\right). \tag{5.2}$$

**c.** Assume that[2] for every $j \neq 0$ we have $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} a_{j+n}\bar{a}_n = 0$. Prove that

$$\lim \frac{1}{N}\sum_{n=1}^{N} a_n = 0. \tag{5.3}$$

**d.** If $\alpha$ is an irrational multiple of $\pi$, the sequence $\{n^2\alpha\}$ is equidistributed on $\mathbb{T}$.

[2] Or for $j$ outside a sequence of density zero in $\mathbb{Z}$.