Math 205C      Homework assignment #4

due on April 29, 1996.

**4.1** Let $f$ be continuous on $[0,1]$. Denote:

$$g(y) = \#f^{-1}(y) = \text{ number of points } x \text{ such that } f(x) = y.$$

Prove that $f$ is of bounded variation $V$ if and only if $g$ is integrable, and then $V = \int g(y)dy$.

**4.2** Let $Y$ be the vector space of all the real-valued functions on $[0,1]$ and let $\tau$ be the weak topology determined by the linear functionals (point valuations) $p_x: f \mapsto f(x), x \in [0,1]$.

**a.** Check that $\tau$ is the topology of pointwise convergence in $[0,1]$.

**b.** Prove that $C([0,1])$ is dense in $(Y,\tau)$.

**c.** Is $C([0,1])$ sequentially dense in $(Y,\tau)$, i.e., is every $f \in Y$ the limit (in the $\tau$ topology) of a sequence $\{f_n\} \subset C([0,1])$?

**4.3** A Peano curve in $\mathbb{R}^n$ is a curve (the image under a continuous map of $[0,1]$ into $\mathbb{R}^n$) that covers a non-empty open set.

Prove that if the sequence $\{\lambda_n\} \subset \mathbb{N}$ grows fast enough, then the curve $\gamma$ defined by $\gamma$: $t \mapsto (x(t), y(t))$, where $t \in [-1,1]$ and

$$x(t) = \sum_1^\infty 10^{2-2n} \cos\lambda_{2n} t, \quad y(t) = \sum_1^\infty 10^{2-2n} \cos\lambda_{2n+1} t,$$

covers the square $S = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$.

*Hints:* Since[1] $\gamma$ is closed it is enough to show that it is $\varepsilon$-dense in the square $S$. Denote

$$x_n(t) = 10^{2-2n} \cos\lambda_{2n} t, \quad y_n(t) = 10^{2-2n} \cos\lambda_{2n+1} t, \quad \gamma_n^*: t \mapsto (x_n(t), y_n(t)),$$

and $\gamma_m: t \mapsto (\sum_1^m x_n(t), \sum_1^m y_n(t))$.

If $\gamma_m$ is $\varepsilon$-dense in $S$, then $\gamma$ is $(\varepsilon + 10^{-2m})$-dense in $S$.

**a.** Show that for any $\varepsilon_n > 0$, by taking $\lambda_{2n+1} >> \lambda_{2n}$, one can insure that $\gamma_n^*$ is $\varepsilon_n$-dense in $[-10^{2-2n}, 10^{2-2n}] \times [-10^{2-2n}, 10^{2-2n}]$.

**b.** Show that, given any $\varepsilon > 0$, if $\lambda_{2n} >> \lambda_{2n-1}$, then (the range of) $\gamma_m$ is $\varepsilon$-close to the (set theoretic sum) $\gamma_{m-1} + \gamma_m^*$.

[1] We use $\gamma$ both as the map and its range.

**4.4** **a.** Let $O \subset \mathbb{R}^2$ be open and bounded. Prove that the Hausdorff dimension of $\mathsf{bdry}(O)$ is at least 1.

**b.** Prove that the set of points covered multiply by a Peano curve (in $\mathbb{R}^2$) has Hausdorff dimension $\geq 1$.

Hint: If $\gamma$ is a Peano curve, covering the connected open set $O \subset \mathbb{R}^2$, and $t_0$ is such that $\gamma([0,t_0])$ covers some but not all of $O$ consider $O \cap \mathsf{bdry}(\gamma)([0,t_0])$

$*$**c.** **(Bonus problem)** Construct an example where the dimension is in fact 1.