Math 205C Homework assignment #1

due on April 8, 2009.

1.1 Let V be a vector space and $\{f_\alpha\}_{\alpha\in A}$ be linear functionals on V. Assume that g is a linear functional on V which is continuous in the weak topology determined by $\{f_\alpha\}_{\alpha\in A}$. Prove that g is in the linear span of $\{f_\alpha\}_{\alpha\in A}$.

1.2 *Hausdorff dimension.* The Hausdorff dimension $\mathscr{H}\text{-dim}(E)$ of a set $E\subset\mathbb{R}$ is the infimum of the numbers c for which there is a constant C such that, for every $\varepsilon>0$, there exists a covering of E by intervals I_n satisfying:

$$\sup_n |I_n| < \varepsilon \quad\text{and}\quad \sum |I_n|^c < C. \tag{1.1}$$

a. Assume that a closed set $E\subset\mathbb{R}$ carries a probability measure μ such that $\mu(I)\le C|I|^\delta$ for every interval I. Prove that $\mathscr{H}\text{-dim}(E)\ge\delta$.

b. What is the Hausdorff dimention of the classical Cantor set

$$E=\{x : x=\sum_1^\infty \varepsilon_j 3^{-j},\ \varepsilon_j=0,2.\} \tag{1.2}$$

1.3 A number $\xi\in\mathbb{R}$ is *diophantine* (with exponent k, $k>0$) if for some constant $C>0$ there are no rational numbers p/q such that $|\xi-p/q|<Cq^{-k}$.

ξ is *a Liouville number* if it is not diophantine, i.e., if for every $k>0$ there exist integers p,q such that $q>k$ and $|\xi-p/q|\le q^{-k}$.

a. Prove that the set $\mathscr{D}\mathit{ioph}$ of diophantine numbers is of the first category; equivalently, the set $\mathscr{L}\mathit{iouv}$ of Liouville numbers is residual in $\mathbb{R}$.

Hint: Write

$$E_k=\bigcap_N \bigcup_{p,q\in\mathbb{Z},\ q>N} \{\xi : |\xi-p/q|<q^{-k}\}$$

so that

$$\mathscr{L}\mathit{iouv}=\bigcap_k E_k\,,$$

clearly a G_δ, a countable intersection of open sets. Verify that it is dense.

b. Prove that the Hausdorff dimension of the set $\mathscr{L}\mathit{iouv}$ is zero.

Hint: Estimate the Hausdorff dimension of E_k.

1.4 Recall that a Banach space B is *uniformly convex* if for any $\varepsilon\in(0,1)$ there exist $\eta<1$ such that if $x,y\in B$, $\|x\|=\|y\|=1$ and $\|x-y\|>2\varepsilon$ then $\|\frac{1}{2}(x+y)\|<\eta$.

1. Check that Hilbert space is uniformly convex, and that one may take $\eta = (1-\varepsilon^2)^{\frac{1}{2}}$.
2. Prove that for $1 < p < \infty$, L^p is uniformly convex.

Hints: For (a) use plane geometry.

For (b): Assume $\int |f|^p d\mu = \int |g|^p d\mu = 1$, and $\int |f-g|^p d\mu = a > 0$.

Write $\phi = \frac{f}{(|f|^p+|g|^p)^{1/p}}$, $\psi = \frac{g}{(|f|^p+|g|^p)^{1/p}}$ and $\nu = (|f|^p + |g|^p)\mu$.

Then $\int |\phi| d\nu = \int |\psi| d\nu = 1$, $\int |\phi - \psi|^p d\nu = a > 0$, and $\int |f+g|^p d\mu = \int |\phi + \psi|^p d\nu$.

This reduces the general case to that where the functions are bounded by 1 and the measure space has total mass 2. Use the convexity of the function x^p in $[-1, 1]$: for $p > 1, 0 < a < 2$, there exists $c = c(p,a) > 0$ such that:

$$\text{if } x_1, x_2 \in [-1,1] \text{ and } |x_2 - x_2| \geq a \text{ then } |\tfrac{x_1+x_2}{2}|^p \leq \tfrac{|x_1|^p+|x_2|^p}{2} - c.$$

Extend this to: if z_1, z_2 are complex numbers,

$$|z_j| \leq 1,\ j = 1,2, \text{ and } |z_2 - z_2| \geq a \text{ then } |\tfrac{z_1+z_2}{2}|^p \leq \tfrac{|z_1|^p+|z_2|^p}{2} - c.$$

If $\int |\phi - \psi|^p d\nu \geq (2\varepsilon)^p$, then $|\phi - \psi| > a$ on a set E of ν measure $> a$, where $a > 0$ depends only on p and ε.

1.5 Let B be a uniformly convex Banach space. Assume that $x_n \in B$ for $n = 1, 2, \ldots$, $x_n \longrightarrow x_0$ in the weak topology and $\|x_n\| \longrightarrow \|x_0\|$. Prove that $x_n \longrightarrow x_0$ in norm.

Give an example showing that the statement is false for general Banach spaces.

1.6 Let $f, g \in L^1(\mathbb{T})$ and assume that for any $\psi \in C^\infty(\mathbb{T})$,

$$\int f(t)\,\psi'(t)\,dt = -\int g(t)\,\psi(t)\,dt$$

(this is often referred to as "g is a *weak derivative* of f") Prove that f is absolutely continuous and $g = f'$ a.e.