1.1 Let $B$ be a Banach space, $\mathcal{L}(B)$ the algebra of bounded (linear) operators on $B$.

a. Prove that if $T \in \mathcal{L}(B)$ is injective, has dense range, and is bounded below (that is $\inf_{\|x\|=1} \|Tx\| > 0$) then $T$ is invertible. Conclude that $T$ fails to be invertible for one of the following mutually exclusive reasons:

- $T$ is not injective, i.e., it has a non-trivial kernel: $\ker(T) = \{ x : Tx = 0 \} \neq \{0\}$.
- $T$ is injective and $TB$ is dense in $B$, but $\inf_{\|x\|=1} \|Tx\| = 0$, $(T$ is not “bounded below”).
- $T$ is injective, but $TB$ is not dense in $B$, that is $\ker(T) = \{0\}$ but $\ker(T^*) \neq \{0\}$.

The spectrum of $T \in \mathcal{L}(B)$ is, by definition, the set $\sigma(T) = \{ \lambda : \exists(T - \lambda I)^{-1} \}$.

b. Prove that $\sigma(T)$ is compact.

Hint: Prove that the set of invertible elements in $\mathcal{L}(B)$ is open.

The point-spectrum $\sigma_p(T)$ is the set of $\lambda$ such that $T - \lambda I$ satisfies s-1 above. Check that $\sigma_p(T)$ consists of the eigenvalues of $T$.

The continuous-spectrum $\sigma_c(T)$ is the set of $\lambda$ such that $T - \lambda I$ satisfies s-2 above. $\sigma_c(T)$ is also referred to as the approximate-point-spectrum and its elements as approximate eigenvalues. (Observe that $\lambda$ is an approximate eigenvalue if for every $\varepsilon > 0$ there exist a unit vector $v$ such that $\|Tv - \lambda v\| < \varepsilon$.)

The residual-spectrum $\sigma_r(T)$ is the set of $\lambda$ such that $T - \lambda I$ satisfies s-3 above.

1.2 Let $T$ be a bounded operator on $B$ and $P(T) = \sum_{n=0}^{N} a_n T^n$ with $a_n \in \mathbb{C}$.

a. Prove: $\sigma(P(T)) = P(\sigma(T))$.

Hint: Write $P(T) - \lambda = a_N \prod_{\lambda_j} (T - \lambda_j)$.

b. Refine: $\sigma_p(P(T)) = P(\sigma_p(T))$ and $\sigma_c(P(T)) \subset P(\sigma_c(T))$.

1.3 Let $\mathcal{H} = l^2(\mathbb{N})$ denote the Hilbert space of one sided square summable numerical sequences $f = \{ a_0, a_1, \cdots, a_n, \cdots \}$, with $\|f\| = \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}$.

The map $S$, defined by $S \{ a_n \} = \{ 0, a_0, a_1, \cdots \}$ is an isometry of $\mathcal{H}$ into itself. Describe and classify its spectrum.

Hint: $\mathcal{H}$ is isometric to the (Hardy) space $H^2$ of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in the unit disc $D = \{ z : |z| < 1 \}$, such that

$$\|f\| = \lim_{r \to 1} \left( \frac{1}{2\pi} \int |f(re^{it})|^2 \, dt \right)^{\frac{1}{2}} = \left( \sum |a_n|^2 \right)^{\frac{1}{2}} < \infty,$$

and $S$ is conjugate to the operator $f \mapsto zf$ on $H^2$.

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For the notation we assume that $N$ is the degree of $P$, i.e., $a_N \neq 0$. 

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**Answer:** The spectrum is the closed unit disc \( \{ \lambda : |\lambda| \leq 1 \} \).

Every \( \lambda \) such that \( |\lambda| < 1 \) is in the residual spectrum. (Verify that if \( |\lambda| < 1 \) the operator \( S - \lambda \) is injective, but its range \( (S - \lambda)\mathcal{H} \) is a closed subspace of codimension 1).

Every \( \varepsilon \) on the boundary (i.e. \( |\varepsilon| = 1 \)) is in the continuous spectrum.

\( (S - \lambda) \) is not bounded below: for \( |\lambda| = 1 \) write \( f_N = \sum_{n=0}^{N} \lambda^n \varepsilon_n \) and observe that \( \|f_N\| = \sqrt{N+1} \) while

\[
(S - \lambda)f_N = \lambda^N \varepsilon^{N+1} - \lambda,
\]

and \( \|(S - \lambda)f_N\| = \sqrt{2} \).

1.4 Prove that the spectrum \( \sigma(T) \) depends continuously on \( T \): For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \|T - T_1\| < \delta \), then every point in \( \sigma(T_1) \) lies within \( \varepsilon \) from \( \sigma(T) \).

Hint: Check that, for some constant \( K \), \( \|(T - \lambda)^{-1}\| \leq K \) on the set \( \Lambda = \{ \lambda : \text{dist}(\lambda, \sigma(T)) \geq \varepsilon \} \).

Take \( \delta < K^{-1} \), then

\[
T_1 - \lambda = (T - \lambda)(1 + (T - \lambda)^{-1}(T_1 - T))
\]

and for \( \lambda \in \Lambda \) both factors are invertible.

1.5 Prove:

a. Every measurable homomorphism of \( \mathbb{T} \) into \( \mathbb{T}^* = \{ \varepsilon : |\varepsilon| = 1 \} \subset \mathbb{C} \) is given by \( t \mapsto e^{int} \) with \( n \in \mathbb{N} \).

**Answer:** The quick answer is to check that the Fourier series of a measurable homomorphism \( \varphi \), well defined since \( \varphi \in L^1(\mathbb{T}) \), consists of a single non zero summand. The “real variables” proof can be done as follows.

Let \( \varphi \) be a measurable homomorphism of \( \mathbb{T} \) into \( \mathbb{T}^* = \{ \varepsilon : |\varepsilon| = 1 \} \).

a.1. Claim: \( \varphi \) is continuous at \( t = 0 \), (and hence everywhere!).

Proof: Given \( \varepsilon > 0 \), divide \( \mathbb{T}^* \) into \( N > \frac{\varepsilon}{\varphi(1)} \) equal arcs \( I_j \). Let \( j \) be such that \( E_j = \varphi^{-1}(I_j) \) has positive measure, and observe that \( \varphi \) maps \( E_j - E_j \) into an arc of length \( < \frac{\varepsilon}{N} \) centered at \( \varepsilon = 1 \). But \( E_j - E_j \) contains an interval \( [-\delta, \delta] \). It follows that if \( |t_1 - t_2| < \delta \), then \( |\varphi(t_1) - \varphi(t_2)| < \varepsilon \).

If \( \varphi(t) = 1 \) for all \( t \) we have \( n = 0 \) and nothing to prove.

Otherwise the range of \( \varphi \) is a dense subgroup of \( \mathbb{T}^* \), and since \( \varphi \) is continuous, the range is all of \( \mathbb{T}^* \). In particular, \( \Gamma = \varphi^{-1}(1) \subset \mathbb{T} \) is a finite subgroup, that is, the group of roots of unity of order \( m \) for some positive integer \( m \).

a.2. For \( t \in (-\pi, \pi] \) write \( \varphi(t) = e^{i\psi(t)} \) with \( \psi \) continuous and \( \psi(0) = 0 \), \( (i\psi(t) \) is a branch of \( \log \varphi(t) \). The assumption that \( \varphi \) is a homomorphism translates to \( \psi(t_1 + t_2) = \psi(t_1) + \psi(t_2) \) when \( t_1, t_2, t_1 + t_2 \) are all in \( [-\pi, \pi] \). In particular \( \psi(k\vartheta) = k\psi(\vartheta) \) for all \( \vartheta \) and \( k \) such that \( k\vartheta \in (-\pi, \pi] \).

a.3. \( \psi \) has no zeros in \( J = (0, \frac{2\pi}{m}) \) and thus have a constant sign. This implies that \( \psi \) is monotone, increasing if it is positive in \( J \), and decreasing otherwise. Write \( n = m \) if \( \psi \) is increasing, \( n = -m \) otherwise.

Thus \( \psi \) is continuous and monotone on \( (-\pi, \pi] \), and \( \psi\left(\frac{2\pi}{m}\right) = \frac{2\pi}{m} \). It follows that \( \psi(t) = nt \) and \( \varphi(t) = e^{int} \).
b. Every measurable homomorphism of $\mathbb{R}$ into $\mathbb{T}^*$ is given by $x \mapsto e^{i\xi x}$ with $\xi \in \mathbb{R}$.

**Answer:** If $\varphi$ is a measurable homomorphism of $\mathbb{R}$ into $\mathbb{T}^*$, it is continuous (same proof as above). If $\varphi(x) = 1$ for all $x$ we have $\xi = 0$. Otherwise, write $\varphi(x) = e^{i\psi(x)}$ with a continuous $\psi$, and check that $\psi$ is linear. (Notice that this is somewhat simpler than the previous case since $\psi$ is now defined and is continuous globally.

Another way is to check that the subgroup $G = \{x: \varphi(x) = 1\}$ is not trivial, which reduces the problem to that on $\mathbb{R}/G$, i.e. to measurable homomorphism on $\mathbb{T}$.

c. Every measurable proper subgroup of $\mathbb{T}$ or of $\mathbb{R}$ has zero Lebesgue measure.

**Answer:** If $G \subset \mathbb{T}$ is a subgroup, then $G - G \subset G$. If $G$ has positive measure, $G - G$ contains an interval; and since every interval spans $\mathbb{T}$, $G = \mathbb{T}$ is not a proper subgroup. Similar proof when $G \subset \mathbb{R}$.

1.6 The spectral norm of $T$, (denoted here by $\|T\|_{\text{sp}}$, another common notation is $\|T\|_{\infty}$) is defined by $\|T\|_{\text{sp}} = \max_{\lambda \in \sigma(T)} |\lambda|$. Prove that $\|T\|_{\text{sp}} = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}$. 

**Answer:** There are two issues: the existence of the limit and its relation to the spectral norm.

a. For the existence of the limit notice that $a_n = \log \|T^n\|$ is subadditive: $a_{n+m} \leq a_n + a_m$.
This implies that $a_{km} \leq ka_n$, or $\frac{1}{kn}a_{km} \leq \frac{1}{n}a_n$, for all $k \in \mathbb{N}$. This, in turn, implies $\limsup_{n} \frac{1}{n}a_n = \lim inf_{n} \frac{1}{n}a_n$.

b. The series $\sum T^n \lambda^{-n}$ converges in norm for $|\lambda| > \lim \|T^n\|^\frac{1}{n}$ so that $\|T\|_{\text{sp}} \leq \liminf_{n \to \infty} \|T^n\|^\frac{1}{n}$. On the other hand, for all $x \in B$, $y \in B^*$, the Laurent expansion $\langle (T - \lambda I)^{-1}x, y \rangle = \sum \langle T^n x, y \rangle \lambda^{-n}$ converges for $|\lambda| > \|T\|_{\text{sp}}$.

This means that, given $x \in B$, the sequence $\langle T^n x, y \rangle \lambda^{-n}$ is bounded for every $y \in B^*$ and the uniform boundedness principle implies that $\|\lambda^{-n} T^n x\|$ is bounded (for every $x \in B$). Applying the uniform boundedness principle once more we have $\lambda^{-n} \|T^n\| = O(1)$. 

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