SOLUTIONS TO THE MIDTERM

5.1 Let $f$ be holomorphic in the upper half disc $D^+ = \{z : |z| < 1, \Im z > 0\}$, and continuous in $\{z : |z| < 1, \Im z \geq 0\}$. Assume that $f$ is real-valued on the real axis, and extend the definition of $f$ to the (open) lower half disc $D^- = \{z : |z| < 1, \Im z < 0\}$ by setting, for $z \in D^-$, $f(z) = \overline{f(z)}$. Prove that the extended $f$ is holomorphic in the unit disc.

Write $f(x + iy) = u(x + iy) + v(x + iy)$, for $x + iy \in D^+$. Then, for $x + iy \in D^-$, we can write $f(x + iy) = U(x + iy) + iV(x + iy) = f(x + iy) = u(x - iy) - iv(x - iy)$. Therefore,

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial V}{\partial y}$$

where we used the fact that $f$ is holomorphic on $D^+$ on the second identity, and similarly,

$$\frac{\partial U}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial V}{\partial x}$$

So, $f$ satisfies the Cauchy-Riemann equations on $D^-$, and hence it is holomorphic in this region.

Since $f$ is continuous in $D^+ \cup (D \cap \mathbb{R})$, and $f(z) = \overline{f(z)}$ for $z \in D^-$, we conclude that $f$ is continuous in $D$. By Morera’s theorem, $f$ is holomorphic in $D$, if

$$\int_{\partial R} f(z)dz = 0$$

for any rectangle $R \subset D$ whose edges are horizontal and vertical. If $R \subset D^+$ or $R \subset D^-$, then the integral vanishes, because $f$ is holomorphic in these regions. Suppose now that $R$ intersects the real line. We will assume that $R$ does not have an edge contained in the real line. That case would also follow from an argument similar to the one we present below. We can decompose $R$ into three rectangles, as in the following picture

![Diagram](attachment:image.png)
Then
\[ \int_{\partial R} f(z)\,dz = \int_{\partial R_1} f(z)\,dz + \int_{\partial R_2} f(z)\,dz + \int_{\partial R_3} f(z)\,dz, \]
and the first and third integrals on the right hand side vanish because \( f \) is holomorphic in \( D^+ \) and \( D^- \). Write \( \partial R_2 = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \) as in the picture

We have that
\[ \int_{\partial R_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz. \]
Since \( R \) is compact, \( \sup_{z \in R} |f(z)| < \infty \). Call this number \( M \). If length(\( \gamma_2 \)) = length(\( \gamma_4 \)) = \( \epsilon \),
\[
\left| \int_{\partial \gamma_2} f(z)\,dz \right| \leq \epsilon M \to 0
\]
as \( \epsilon \to 0 \), and similarly
\[
\left| \int_{\partial \gamma_4} f(z)\,dz \right| \leq \epsilon M \to 0.
\]
Now, parametrizing \( \gamma_1 \) by \( z = x - i \frac{\epsilon}{2}, x \in [a, b] \) and \( -\gamma_3 \) (i.e., \( \gamma_3 \) with the reverse orientation) by \( z = x + i \frac{\epsilon}{2}, x \in [a, b] \),
\[
\int_{\partial \gamma_1} f(z)\,dz + \int_{\partial \gamma_3} f(z)\,dz = \int_{a}^{b} f \left( x - i \frac{\epsilon}{2} \right)\,dx + \int_{a}^{b} f \left( x + i \frac{\epsilon}{2} \right)\,dx =
\]
\[
= \int_{a}^{b} f \left( x - i \frac{\epsilon}{2} \right) - f \left( x + i \frac{\epsilon}{2} \right)\,dx.
\]
Since \( f \) is continuous in \( D \) and \( R \) is compact, \( f \) is uniformly continuous in \( R \), and so given any \( \delta > 0 \), we can choose \( \epsilon > 0 \) such that \( |f(x - i \frac{\epsilon}{2}) - f \left( x + i \frac{\epsilon}{2} \right)| < \delta \) for all \( a < x < b \).
Therefore,
\[
\left| \int_{a}^{b} f \left( x - i \frac{\epsilon}{2} \right) - f \left( x + i \frac{\epsilon}{2} \right)\,dx \right| < (b - a)\delta,
\]
and this can be made arbitrarily small. We conclude that \( \int_{\partial R_2} f(z)\,dz \) is arbitrarily small, so it vanishes, as we wanted to show.

5.2 For \( f \) meromorphic in a region \( \Omega \) and \( z_0 \in \Omega \) let \( \mathrm{Res}(f, z_0) \) denote the residue of \( f \) at \( z_0 \).
\[ \mathrm{a.} \text{ Let } f = \frac{g}{h}, \text{ where } g \text{ and } h \text{ are holomorphic in } \Omega, \text{ and } h \text{ has a simple zero at } z_0 \in \Omega. \]
Prove that \( \mathrm{Res}(f, z_0) = \frac{g'(z_0)}{h''(z_0)} \).
Since $h$ has a simple zero at $z_0$, $h(z) = (z - z_0)h_1(z)$ for some function $h_1$ that is holomorphic and non-vanishing near $z_0$. So, $(z - z_0)f(z)$ has a removable singularity at $z_0$, and admits a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ near $z_0$. Hence, $f(z) = \frac{1}{z - z_0} \sum_{n=0}^{\infty} a_n(z - z_0)^n$ near $z_0$, and

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z) = \lim_{z \to z_0} \frac{g(z)}{h_1(z)} = \frac{g(z_0)}{h_1(z_0)}.$$  

Since $h'(z) = h_1(z) + (z - z_0)h_1'(z)$, we have $h'(z) = h_1(z_0)$, and the desired conclusion follows.

b. Assume that $F$ has a pole of order $m$ at $z_0$. Prove that

$$\text{Res}(F, z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m F(z)]_{z=z_0}.$$  

Near $z_0$, we can write $F(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n$, and $\text{Res}(F, z_0) = a_{-1}$. But

$$\frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m F(z)] = \frac{d^{m-1}}{dz^{m-1}} \sum_{n=-m}^{\infty} a_n(z - z_0)^{n+m} = \sum_{n=-m}^{\infty} \frac{(n + m)!}{(n + 1)!} a_n(z - z_0)^{n+1}$$  

Evaluating this expression at $z = z_0$, we get $(m - 1)!a_{-1}$, and the result follows.

c. What is $\text{Res}(z^{-7}e^z, 0)$?

Since $e^{0} \neq 0$, $\frac{e^z}{z^7}$ has a pole of order 7 at 0. So, by the previous exercise,

$$\text{Res} \left( \frac{e^z}{z^7}, 0 \right) = \frac{1}{6!} \frac{d^6}{dz^6} \left[ e^z \right]_{z=0} = \frac{1}{720}.$$  

5.3 Prove:

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{1 + x^2} = \pi,$$

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\cos x \, dx}{1 + x^2} = \frac{\pi}{e}.$$  

The meromorphic function $\frac{1}{1+z^2} = \frac{1}{2i(z-i)} - \frac{1}{2i(z+i)}$ has a simple pole at $i$ and $-i$. Let $\gamma$ be a piecewise $C^1$ curve around $i$, oriented counterclockwise, and consisting of a line segment $\gamma_1$ along the real axis, from $-R$ to $R$, and a half circle $\gamma_2$ in the upper half plane, centered at 0 and with radius $R$. Then, by the residue theorem,

$$\int_{\gamma} \frac{1}{1+z^2} \, dz = 2\pi i \text{Res} \left( \frac{1}{1+z^2}, i \right) = \pi$$  

and

$$\int_{\gamma} \frac{1}{1+z^2} \, dz = \int_{\gamma_1} \frac{1}{1+z^2} \, dz + \int_{\gamma_2} \frac{1}{1+z^2} \, dz = \int_{-R}^{R} \frac{1}{1 + x^2} \, dx + \int_{\gamma_2} \frac{1}{1+z^2} \, dz$$  

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Therefore, we only need to show that \( \lim_{R \to \infty} \int_{\gamma_2} \frac{1}{1 + z^2} \, dz = 0 \). Parametrize \( \gamma_2 \) by \( z(\theta) = R e^{i \theta}, \theta \in [0, \pi] \). Then, for \( R > 1 \),
\[
\left| \int_{\gamma_2} \frac{1}{1 + z^2} \, dz \right| = \left| \int_0^\pi \frac{i R e^{i \theta}}{1 + R^2 e^{2i \theta}} \, d\theta \right| \leq \left| \int_0^\pi \frac{R}{R^2 - 1} \, d\theta \right| = \frac{\pi R}{R^2 - 1} \xrightarrow{R \to \infty} 0
\]
as we wanted to show.

For the second integral in the integral, recall first that \( e^{iz} = \cos z + i \sin z \). As before, the meromorphic function \( \frac{e^{iz}}{1 + z^2} = \frac{e^{iz}}{2(\pi i - i)} - \frac{e^{iz}}{2(\pi i + i)} \) has a simple pole at \( i \) and \(-i\). Taking \( \gamma = \gamma_1 \cup \gamma_2 \) as previously,
\[
\int_{\gamma} \frac{e^{iz}}{1 + z^2} \, dz = 2 \pi R \text{Res} \left( \frac{e^{iz}}{1 + z^2}, i \right) = \frac{\pi}{e}.
\]
and
\[
\int_{\gamma_2} \frac{e^{iz}}{1 + z^2} \, dz = \int_{-R}^{R} \frac{e^{ix}}{1 + x^2} \, dx + \int_{\gamma_2} \frac{e^{iz}}{1 + z^2} \, dz
\]
Parametrizing \( \gamma_2 \) as before,
\[
\left| \int_{\gamma_2} \frac{e^{iz}}{1 + z^2} \, dz \right| = \left| \int_0^\pi \frac{e^{i R e^{i \theta}} \sin \theta}{1 + R^2 e^{2i \theta}} \, d\theta \right| \leq \left| \int_0^\pi \frac{e^{-R \sin \theta}}{R^2 - 1} \, d\theta \right| \leq \left| \int_0^\pi \frac{R}{R^2 - 1} \, d\theta \right| = \frac{\pi R}{R^2 - 1} \xrightarrow{R \to \infty} 0
\]
where we used the fact that \( \sin \theta \geq 0 \), for \( 0 \leq \theta \leq \pi \). Therefore,
\[
\lim_{R \to \infty} \int_{-R}^{R} \frac{\cos x \, dx}{1 + x^2} = \mathfrak{R} \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{1 + x^2} \, dx = \mathfrak{R} \lim_{R \to \infty} \int_{\gamma} \frac{e^{iz}}{1 + z^2} \, dz = \frac{\pi}{e}
\]

5.4 Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and assume that the radius of convergence \( R \) of the series is positive and finite. For every \( \theta \in [0, 2\pi] \) denote by \( r_\theta \) the radius of convergence of the Taylor expansion of \( f \) around \( z_0 = \frac{e^{i \theta}}{2} \).

a. Prove that \( r_\theta \) is a continuous function of \( \theta \).

We show that for any \( \theta_1, \theta_2 \in [0, 2\pi] \), \( |r_\theta - r_{\theta_2}| \leq \frac{R}{2} |\theta_1 - \theta_2| \), which implies that \( r_\theta \) is continuous. Since
\[
|z_{\theta_1} - z_{\theta_2}| = \frac{R}{2} \left| e^{i(\theta_1 - \theta_2)} - 1 \right| = \frac{R}{2} \left| \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \right| \leq \frac{R}{2} |\theta_1 - \theta_2|
\]
it is enough to prove that \( |r_{\theta_1} - r_{\theta_2}| \leq |z_{\theta_1} - z_{\theta_2}| \). Suppose otherwise that \( |r_{\theta_1} - r_{\theta_2}| > |z_{\theta_1} - z_{\theta_2}| \) for some \( \theta_1, \theta_2 \). Without loss of generality, we can assume \( r_{\theta_1} - r_{\theta_2} > |z_{\theta_1} - z_{\theta_2}| \). But then, there is a disc centered at \( z_{\theta_2} \) of radius greater than \( r_{\theta_2} \), which is contained in \( D(z_{\theta_1}, r_{\theta_1}) \). Since \( f \) can be extended to a holomorphic function on \( D(0, R) \cup D(z_{\theta_1}, r_{\theta_1}) \), this contradicts the assumption that the radius of convergence of the power series centered at \( z_{\theta_2} \) is \( r_{\theta_2} \).
b. Prove that $\min_{\theta} r_\theta = R/2$, and $\max_{\theta} r_\theta \leq 3R/2$.

For any $\theta \in [0, 2\pi]$, $D(z_\theta, R/2) \subset D(0, R)$, and $f$ is holomorphic on the latter region. Therefore, $\min_{\theta} r_\theta \geq R/2$. On the other hand, if $\min_{\theta} r_\theta > R/2$, then one could extend $f$ to a holomorphic function on a disc centered at 0 of radius $R/2 + \min_{\theta} r_\theta > R$, which contradicts the fact that $R$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

If $\max_{\theta} r_\theta > 3R/2$, let the maximum be attained at $\theta_0$. Then, $f$ would have a holomorphic extension to $D(z_{\theta_0}, \max_{\theta} r_\theta)$, which would contain a disc centered at 0 with radius $\max_{\theta} r_\theta - R/2 > R$. This would again contradict the fact that $R$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

c. Prove that if $a_n \geq 0$ for all $n$ then $r_0 = R/2$.

Proof 1: Since
$$\frac{d^k}{dz^k} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$
the Taylor series of $f$ around $R/2$ is given by
$$\sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \frac{n!}{(n-k)!k!} a_n \left( \frac{R}{2} \right)^{n-k} \right) \left( z - \frac{R}{2} \right)^k = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} a_n \left( \frac{R}{2} \right)^{n-k} \right) \left( z - \frac{R}{2} \right)^k$$

Suppose $r_0 > R/2$. Then, this power series converges at $z = R + \epsilon$, for some $\epsilon > 0$. Therefore,
$$\sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} a_n \left( \frac{R}{2} \right)^{n-k} \right) \left( \frac{R}{2} + \epsilon \right)^k = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} a_n \left( \frac{R}{2} \right)^{n-k} \left( \frac{R}{2} + \epsilon \right)^k = \sum_{n=0}^{\infty} a_n (R + \epsilon)^n < \infty$$
where we used the fact that the $a_n$ are non-negative real numbers when we changed the order of the terms in the series. We now reached a contradiction with the assumption that the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is $R$.

Proof 2: We again use the fact that
$$\frac{d^k}{dz^k} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$
Given $z_\theta = R/2 e^{i\theta}$, the Taylor series centered at $z_\theta$ is
$$\sum_{k=0}^{\infty} \frac{d^k}{dz^k} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$
$$\sum_{k=0}^{\infty} \binom{n}{k} a_n (z - z_\theta)^k = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} a_n z^{n-k} \right) (z - z_\theta)^k$$
Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be an entire function. Denote \( U(z) = \Re f(z) \) and \( V(z) = \Im f(z) \), and let \( A(r) = \max_{|z| \leq r} |U(z)| \) and \( M(r) = \max_{|z| \leq r} |f(z)| \). Clearly \( A(r) \leq M(r) \) and it is known that the ratio \( M(r)/A(r) \) is unbounded in general.

However, prove Borel-Caratheodory inequality: If \( r < R \), then \( M(r) \leq \frac{2r}{R-r} A(R) \).

Write \( \alpha_n = \Re a_n \) and \( \beta_n = \Im a_n \). Then,

\[
U(Re^{i\theta}) = \Re \left( \sum_{n=1}^{\infty} (\alpha_n + i\beta_n)R^n (\cos n\theta + i\sin n\theta) \right) = \sum_{n=1}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) R^n
\]

Now, note that

\[
\int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \pi \delta_{mn} = \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta
\]

for any \( n, m \geq 0 \) (this can be checked using either integration by parts, or the fact that \( \int_0^{2\pi} e^{in\theta} \, d\theta = 2\pi \delta_{n0} \)). These formulas imply that

\[
\int_0^{2\pi} U(Re^{i\theta}) \cos n\theta \, d\theta = \int_0^{2\pi} \sum_{m=1}^{\infty} (\alpha_m \cos m\theta - \beta_m \sin m\theta) R^m \cos n\theta \, d\theta =
\]

\[
= \sum_{m=1}^{\infty} \int_0^{2\pi} (\alpha_m \cos m\theta - \beta_m \sin m\theta) R^m \cos n\theta \, d\theta = \pi \alpha_n R^n
\]

where we used the fact that the power series converges absolutely, hence uniformly on the compact set \( \{ |z| = R \} \), and therefore one can commute the sum with the integral. Similarly,

\[
\pi \beta_n R^n = - \int_0^{2\pi} U(Re^{i\theta}) \sin n\theta \, d\theta,
\]

hence:

\[
a_n R^n = \frac{1}{\pi} \int_0^{2\pi} U(Re^{i\theta}) e^{-in\theta} \, d\theta \quad \text{implies} \quad |a_n| R^n \leq \frac{1}{\pi} \int_0^{2\pi} |U(Re^{i\theta})| \, d\theta \leq 2A(R).
\]

So, if \( |z| \leq r \),

\[
|f(z)| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \leq \sum_{n=1}^{\infty} |a_n| r^n \leq \sum_{n=1}^{\infty} |a_n| R^n \left( \frac{r}{R} \right)^n \leq 2A(R) \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n = \frac{2A(R) r^n}{1 - \left( \frac{r}{R} \right)}
\]

and

\[
\left| b_k^0 \right| = \left| \sum_{n=k}^{\infty} \frac{n!}{(n-k)! k!} a_n z^{n-k} \right| \leq \sum_{n=k}^{\infty} \frac{n!}{(n-k)! k!} |a_n z^{n-k}| = \left| b_k^0 \right|
\]

(we use the fact that the \( a_n \) are real non negatives in the inequality). Since the radius of convergence of a power series \( \sum_{n=1}^{\infty} a_n z^n \) is \( \left( \lim_{n \to \infty} \sqrt[1/2]{|a_n|} \right)^{-1} \), the fact that \( |b_k^0| \leq |b_k^0| \) implies that \( r_0 \geq r_0 > \frac{R}{2} \), which contradicts the previous exercise.

\[\]
as we wanted to see.