Denote by $MVP(\Omega)$ the space of continuous functions $\varphi$ defined on the region $\Omega$ that have the mean-value property: for all discs $\{z: |z - z_0| \leq r\} \subset \Omega$,

$$\frac{1}{2\pi} \int_{0}^{2\pi} h(z_0 + re^{i\theta})d\theta = h(z_0).$$

a. Prove the maximum modulus principle for functions in $MVP$.

b. Prove that a continuous function $\varphi$ defined on the region $\Omega$ has the mean-value property if and only if it is harmonic.

Hint: If $z_0 \in \Omega$ and $\{z: |z - z_0| \leq r\} \subset \Omega$, compare $\varphi$ to the Poisson integral defined by the restriction of $\varphi$ to the circle $\{z: |z - z_0| = r\}$. (The two agree on the circle.)

c. Let $h_n$, $n \in \mathbb{N}$ be harmonic in a region $\Omega$. Assume that $h_n \to g$ compactly. Prove that $g$ is harmonic in $\Omega$.

Let $F$ be a holomorphic function whose range is contained in a region $\Omega$, and let $h$ be harmonic in $\Omega$. Prove that $h \circ F$ is harmonic.

### (Harnack’s Principle.)

a. Let $h$ be a positive harmonic function in a region containing the closed disc $\overline{D} = \{z: |z| \leq 1\}$. Prove that if $z = re^{i\theta}$, $r < 1$, then

$$\frac{1 - r}{1 + r} h(0) \leq h(z) \leq \frac{1 + r}{1 - r} h(0).$$

Hint: Use the Poisson integral.

b. Let $h_n$ be harmonic in $D$, continuous on $\overline{D}$ and assume that for all $n$, $h_{n+1} - h_n \geq 0$ in $D$. Assume also that $h_n(0)$ converge to a finite limit. Prove that $h_n(z)$ converge uniformly in every disc $\{z: |z| \leq r\}$ of radius $r < 1$.

### Subharmonic functions

**Definition:** A continuous real-valued function $\varphi$ defined in a region $\Omega$ is **subharmonic** if for every $z_0$ and $r > 0$ such that $\overline{D}(z_0, r) = \{z: |z - z_0| \leq r\} \subset \Omega$, we have:

$$\varphi(z_0) \leq \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(z_0 + re^{it})dt.$$

a. Prove: (Maximum principle for $\varphi - h$). If $\varphi$ is subharmonic in $\Omega$, $h$ is real-valued harmonic in $\overline{\Omega}_1 \subset \Omega$, and $\varphi - h$ has a local maximum at an interior point of $\Omega_1$, then $\varphi - h = \text{const}$ on $\Omega_1$.

b. Prove that a continuous function $\varphi$ is subharmonic if and only if, for every $\overline{D}(z_0, r)$ as above and any function $h$ that is real-valued, continuous on $\overline{D}(z_0, r)$ and harmonic in
$D(z_0, r)$, the inequality $\varphi(z) \leq h(z)$ on the boundary of $\overline{D}(z_0, r)$, implies $\varphi(z) \leq h(z)$ in $D(z_0, r)$. In particular, for $z = pe^{i\theta}$ write $\rho_1 = \rho/r$ and:

\[
\varphi(z) \leq \frac{1}{2\pi} \int_0^{2\pi} P(\rho_1, s - t) \varphi(z_0 + re^{it}) dt.
\]  

($P(\cdot, \cdot)$ denotes the Poisson kernel.)

c. Assume that $\varphi$ and $\psi$ are subharmonic in $\Omega$. Prove that the function $\Phi$ defined by $\Phi(z) = \max(\varphi(z), \psi(z))$ for $z \in \Omega$ is subharmonic.

7.5 a. Identify the harmonic function $h_a$ in the annulus $A_a = \{ z : a < |z| < 1 \}$ with boundary values: $h_a(e^{it}) = 0$ and $h_a(ae^{it}) = 1$.

Hint: Verify that $h_a$ is a radial function.

b. Prove that there is no harmonic function on the punctured disc $D_0 = \{ z : 0 < |z| < 1 \}$ with boundary values 1 for $z = 0$ and 0 for $|z| = 1$.

Hint: Check that such function would be $\lim_{a \to 0} h_a$.

7.6 Phragmén-Lindelöf (special case.) Let $f$ be a holomorphic function in the strip

\[
\{ z = x + iy : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \}.
\]

Assume that $|f(z)| \leq m$ on the boundary of the strip, and assume that there is a constant $c \in (0, 1)$ such that $|f(z)| = O(e^{c\cosh cx})$ inside the strip. Prove that $|f(z)| \leq m$.

Hints:

1. Observe that $e^z$ and $\cosh z$ are purely imaginary when $y = \pm \frac{\pi}{2}$. (This shows that $e^{c\cosh z}$ is bounded on the boundary of the strip but not inside, showing that the growth condition given in the problem is close to optimal.)

2. Apply the maximum principle to the functions $f(z)e^{-\varepsilon_n \cosh bx}$, where $c < b < 1$ and $\varepsilon_n$ small positive.