4.1 Assume that \( f \) is continuous in \( D = \{ z : |z| < 1 \} \) and holomorphic in the complement of \( D \cap \mathbb{R} \) in \( D \). Prove that \( f \) is holomorphic in \( D \).

4.2 If \( f_n, n = 1, 2, \ldots \) are holomorphic, never vanishing in a region \( \Omega \), and \( f_n \to F \) compactly, then either \( F(z) \neq 0 \) for all \( z \in \Omega \), or \( F(z) = 0 \) identically.

Hint: Verify and use the fact that on any compact set \( E \subset \Omega \) on which \( F(z) \neq 0 \) and such sets abound unless \( F = 0 \) identically.

\[
\frac{f'_n}{f_n} \to \frac{F'}{F} \quad \text{uniformly.}
\]

4.3 **Schwarz's lemma.** Let \( f \) be holomorphic in \( \{ z : |z| < R \} \). Assume that \( f(0) = 0 \) and \( \sup_{|z|<R}|f(z)| \leq M \).

a. Prove that either \( f(z) = az \), with constant \( a \) (such that \( |a| \leq M/R \)) or

\[
|f(z)| < \frac{M}{R} |z| \quad \text{for} \quad 0 < |z| < R.
\]

Hint: Use the maximum principle.

b. Prove that \( |f'(0)| \leq M/R \).

4.4 \( P(z) = z^n + \sum_{j=1}^{n-1} a_j z^j, |a_j| \leq a \). For what \( R \) can you guarantee that all the zeros of \( P \) are in \( \{ z : |z| < R \} \)?

4.5 Let \( f \) and \( g \) be holomorphic in \( D = D(0, 1) \). Assume that \( z_j, j = 1, \ldots, k \) is an enumeration of the zeros of \( f \) in \( D \), and that \( f \neq 0 \) on \( C \), the boundary of \( D \). Compute

\[
\frac{1}{2\pi i} \int_C \frac{f'(\zeta)}{f(\zeta)} g(\zeta) d\zeta.
\]

4.6 **The winding number of a curve with respect to a point (aka the index of a point with respect to a curve).**

Let \( \gamma \) be a piecewise differentiable closed curve in \( \mathbb{C} \), parametrized by \( \varphi(t), \ 0 \leq t \leq 1 \), and \( z_0 \in \mathbb{C} \setminus \gamma \). Prove that \( n(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\gamma - z_0} \) is an integer.

Hint: Write \( h(t) = \int_0^t \frac{\varphi'(s)}{\varphi(s) - z_0} ds \) so that \( h'(t) = \frac{\varphi''(t)}{\varphi(t) - z_0} \), and check that \( e^{-h(t)} \frac{\varphi(t) - z_0}{\varphi(0) - z_0} \) is a constant. Checking for \( t = 0 \), the constant is 1, and since \( \varphi(0) = \varphi(1), e^{h(1)} = 1 \) and \( h(1) \) is an integral multiple of \( 2\pi i \).