Math 116  Homework assignment #2

due on January 22, 2010.

2.1  Prove (Lucas theorem): The zeros of the derivative of a polynomial $P(z) = \sum a_n z^n$, $a_n \in \mathbb{C}$, are contained in the convex hull of the zeros of $P$.

Remark: The convex hull of a set $E$ is the smallest convex set that contains $E$; it is the intersection of all the half-planes that contain $E$. Reduce the statement of the theorem to the following, (and prove it):

Every half-plane that contains all the zeros of $P$ contains all the zeros of $P'$.

2.2  a.  Let $\gamma$ be a closed contour parametrized by $r(t)e^{it}$, $0 \leq t \leq 2\pi$ and $r(t)$ positive, continuously differentiable on $0 \leq t \leq 2\pi$, and $r(2\pi) = r(0)$. Compute $\int_{\gamma} \zeta^n d\zeta$ for all $n \in \mathbb{Z}$.

Hint: Check that the value obtained is independent of the choice of $r$ so that one may take $r = 1$.

b.  Compute $\int_{|z|=1} e^z z^n dz$, $n \in \mathbb{Z}$.

2.3  Let $f$ be holomorphic in $|z| < 2$, $|z_j| \leq 1$, $j = 1, 2$, $\gamma$ the circle $\{\zeta: |\zeta| = 2\}$ with the positive parameterization. Prove:

$$f(z_1) - f(z_2) = \frac{z_1 - z_2}{2\pi i} \int_{\gamma} \frac{f(\zeta)d\zeta}{\zeta^2 - (z_1 + z_2)\zeta + z_1z_2}.$$  

2.4  Let $\phi_n$ be holomorphic in the unit disc $D(0,1) = \{\zeta: |\zeta| < 1\}$ and assume that, as $n \to \infty$, $\phi_n(z)$ converges to a function $\psi(z)$ and the convergence is uniform in $D(0,r) = \{\zeta: |\zeta| \leq r\}$ for every $0 \leq r < 1$.

Prove that $\psi$ is holomorphic in the open unit disc.

2.5  Let $g(\zeta)$ be continuous complex-valued function on the unit circle $C = \{\zeta: |\zeta| = 1\}$. Write

$$f(z) = \int_C \frac{g(\zeta)d\zeta}{\zeta - z}.$$  

Prove that $f$ is holomorphic in the open unit disc.

2.6  a.  Let $F$ be an entire function (i.e., defined and holomorphic everywhere in $\mathbb{C}$). Prove that if $F$ is not a constant then its range is dense in $\mathbb{C}$.

b.  Let $F$ be an entire function whose range contains no negative real number. Prove that $F$ is a constant.

2.7  Let $f$ be holomorphic in the unit disc, and assume $|f(z)| \leq (1 - |z|)^{-1}$ on $\{z: |z| < 1\}$.

Prove that $f^{(n)}(0) = O((n+1)!)$ (the notation $a_n = O(b_n)$ with capital $O$ means: $a_n \leq K b_n$ for an appropriate constant $K$).

What estimate obtains if $|f(z)| \leq (1 - |z|)^{-3}$ on $\{z: |z| < 1\}$?