Math 114, Spring 2009, Review problems.

1. Let $(\mathscr{V}, T)$ be a linear system over a field $\mathbb{F}$.

a. Define "$(\mathscr{V}, T)$ is semisimple".

b. Prove: If the underlying field $\mathbb{F}$ is algebraically closed and $(\mathscr{V}, T)$ is semisimple, then T is diagonalizable.

2. Given: $\mathscr{H}$ is a complex n-dimensional inner-product space, $S \in \mathscr{L}\mathscr{H}$.

a. Prove that if S is self-adjoint then the orhtogonal complement of every S-invariant subspace is S-invariant.

b. Prove that S is normal if, and only if, the orhtogonal complement of every S-invariant subspace is S-invariant.

c. Use induction, part **a.**, and the fact that every operator has at least one eigenvector to prove the spectral theorem for selfadjoint operators.

d. Extend the proof of **d.** to sets $\mathscr{S}$ of commuting selfadjoint operators.

3. **a.** What does the fact that the matrix of an operator T relative to a basis $\{w_1, \dots, w_n\}$ is lower triangular tell you about T-invariant subspaces?

b. Describe a (unitary) matrix B such that if A is an upper triangular matrix then $B^{-1}AB$ is lower triangular and vice versa.

c. Prove that every matrix $A \in \mathscr{M}(n, \mathbb{C})$ is unitarily equivalent to a lower triangular matrix. Is every matrix $B \in \mathscr{M}(n, \mathbb{R})$ similar to a lower triangular matrix?

4. Let $T \in \mathscr{L}(V)$, $l \in \mathbb{N}$.

a. Prove that $\ker(T^l) = \ker(T^{l+1})$ if, and only if $\mathrm{range}(T^l) = \mathrm{range}(T^{l+1})$.

b. Prove that $\ker(T^l) = \ker(T^{l+1})$ implies $\ker(T^l) = \ker(T^{l+2})$

5. ***Spectral Mapping Theorem:*** Let $\mathscr{V}$ be a complex vector space, $T \in \mathscr{L}\mathscr{V}$, and P a polynomial with complex coeffcients. Then

$$\sigma(P(T)) = \{P(\lambda) : \lambda \in \sigma(T)\}.$$

6. Let $\{v_1, \dots, v_n\} \subset \mathbb{C}^n$ be the rows of the square matrix B. Prove *Hadamard's inequality*:

$$|\det B| \leq \prod \|v_j\| \tag{0.1}$$

7. **Unitary matrices.** A matrix $A \in \mathscr{M}(n, \mathbb{C})$ is unitary if $A^*A = I$.

a. Check that $A^*A = I$ implies $AA^* = I$.

b. A is unitary if, and only if, its rows form an orthonormal set.

c. If A is unitary and B is obtained from A by a permutation of its rows, the B is unitary.

d. If A is unitary and C is obtained from A by a permutation of its columns, the C is unitary.

8. **Unitary operators.** Prove that the following conditions on an operator $T \in \mathscr{L}(\mathscr{H})$ are equivalent: (T is *unitary* if it satisfies any (hence all) of theses conditions.)

a. T is an isometry, that is, $\|Tx\| = \|x\|$ for every $x \in \mathscr{H}$.

b. T preserves the inner product, that is, $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in \mathscr{H}$.

c. T is invertible and its inverse is T^* (its adjoint).

9. Let T be a unitary operator on U,

a. If $W \subset U$ is T-invariant then $T|_W$, the restriction of T to W, is unitary (with respect to the induced inner product).

b. $W^\perp$ is T-invariant.

c. **The spectral theorem for unitary operators.** There exists an orthonormal basis $\{w_1, \dots, w_n\}$ for U such that each w_j is an eigenvector for T.

d. (Rephrasing c. above:) The matrix of T with respect to any orthonormal basis is unitarily equivalent to a diagonal matrix.

10. Let $T \in \mathscr{L}\mathscr{V}$. Show that there exist vectors $v \in \mathscr{V}$ such that $\mathsf{minP}_{T,v} = \mathsf{minP}_T$.

Hints: Show and use:

a. If $\mathsf{minP}_T = \Phi^m$, with Φ irreducible, then there exist vectors $v \in \mathscr{V}$ such that $\mathsf{minP}_{T,v} = \mathsf{minP}_T$.

b. If v_1, $v_2 \in \mathscr{V}$ and minP_{T,v_1} and minP_{T,v_2} are relatively prime, then

$$\mathsf{minP}_{T,v_1+v_2} = \mathsf{minP}_{T,v_1}\,\mathsf{minP}_{T,v_2}.$$

11. Prove: A linear system $(\mathscr{V}, T)$ is cyclic if, and only if, $\mathsf{minP}_T = \chi_T$.

12. For an $n \times n$ matrix $A = (a_{i,j})$, define $\mathsf{trace}A$, and prove that similar matrices have the same trace.

Hint: The characteristic polynomial χ_A satisfies:

$$\chi_A(\lambda) = \prod_{i=1}^{n}(a_{i,i} - \lambda) + Q(\lambda)$$

where Q is a polynomial of degree bounded by $n-2$.

13. **Nilpotent operators:** An operator T is *nilpotent* if $T^k = 0$ for some integer k, the smallest such k is *the height of* T. The *height of a vector* v (relative to T) is, by definition, the smallest integer l such that $T^l v = 0$.

Assume $T \in \mathscr{L}\mathscr{V}$ is nilpotent.

a. Show that if W is a T-invariant subspace, then the operator $\widetilde{T}$ induced by T on V/W is nilpotent, and the height of $\widetilde{T}$ is not bigger than that of T.

b. Let $v_1 \in V$ be of height k, i.e., $T^k v = 0$, and $T^{k-1}v \neq 0$. Prove that the vectors $\{T^j v_1\}_0^{k-1}$ are linearly independent, and their span V_1 is T-invariant. What is the matrix of $T|_{V_1}$ relative to the basis $u_j = T^{j-1}v \;\; j = 1, \dots, k$?

c. Assume that the value of k in the previous part is the height of T. Let $\tilde{w} \in V/V_1$ be of height l (for $\widetilde{T}$). This means that if $w \in V$ is a representative of $\tilde{w}$, (i.e., $\tilde{w} = w + V_1$), then $T^l w \in V_1$. Show that w can be chosen so that $T^l w$ is in fact the zero element of V.

   Hint: Check that if $T^l w = \sum a_j T^j v_1$ then $a_j = 0$ for $j \leq l$, (the height of $\sum_m a_j T^v v_1,\ a_m \neq 0$ is $k-m$), and replace w by $w - \sum a_j T^{j-l} v_1$.

d. Assume again that k above is the height of T. If $k < n$, denote by k_2 the height of $\widetilde{T}|_{(V/V_1)}$.

   (a) If $k_2 = 0$, that is $\widetilde{T}|_{(V/V_1)} = 0$ choose any basis $\{\tilde{z}_1, \dots, \tilde{z}_{n-k}\}$ for V/V_1, Take $z_j \in \tilde{z}_j$, i.e., such that $\tilde{z}_j = z_j + V_1$, and check that $\{T^j v\}_0^{k-1} \cup \{z_j\}_{j=1}^{n-k}$ is a basis for V. Describe the matrix of T relative to this basis.

   (b) If $k_2 \neq 0$, let $\tilde{v}_2 \in V/V_1$ be a vector of height k_2 and $v_2 \in V$ such that $\tilde{v}_2 = v_2 + V_1$ and $T^{k_2} v_2 = 0$ (see part c.). Denote the span of $\{T^j v_2\}_0^{k_2-1}$ by V_2. Prove that $V_1 \cap V_2 = 0$, and repeat the procedure for $T|_{\left(V/(V_1 \oplus V_2)\right)}$.

14. Denote by $\mathsf{S_n}$ the group of permutations of $\{1, \dots, n\}$. Let $\sigma \in \mathsf{S_n}$, and A_σ the $n \times n$ operator which maps e_i onto $e_{\sigma(i)}$. Describe the spectrum of A_σ in terms of the cycle decomposition of σ.

15. Let $P \in \mathbb{F}[x]$ be an irreducible polynomial of degree n, and let $A \in \mathscr{M}(n,\mathbb{F})$ be its companion matrix. Prove that the algebra generated by A is a field.

16. **The spectral theorem:** Assume that T is selfadjoint on $\mathscr{H}$.

a. Prove that $\sigma(T) \subset \mathbb{R}$.

b. Prove that if $(T-\lambda)^k v = 0$ for some k then $(T-\lambda)v = 0$.

c. What is the restriction of T to $\mathscr{H}_\lambda$? ($\mathscr{H}_\lambda$ is defined, as above, as the set $\{u \in \mathscr{H} : \exists k \text{ such that} (T-\lambda)^k u = 0\}$. By the previous part $\mathscr{H}_\lambda = \ker(()T-\lambda).$)

d. If $\lambda_1 \neq \lambda_2$ then $\mathscr{H}_{\lambda_1} \perp \mathscr{H}_{\lambda_2}$.

e. (The spectral theorem) $T = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda$, P_λ being the orthogonal projection on $\mathscr{H}_\lambda$.

17. Prove:

a. $(\mathscr{V}, T)$ is cyclic if, and only if, $\mathrm{minP_T} = \chi_T$.

b. Prove that $\mathrm{minP_T} = \mathrm{minP_{T,\mathit{v}}}$ for some $v \in \mathscr{V}$.

18. (*S*chur's lemma:) Let $\mathscr{H}$ be a complex vector space. A set $\mathscr{S} \subset \mathscr{L}\mathscr{V}$ is *minimal* if no nontrivial subspace of $\mathscr{H}$ is invariant under every $S \in \mathscr{S}$. Prove: $T = \lambda I$ for some complex number λ. *hints:* $\sigma(T)$ is a single point, T is semisimple.

19. Denote by $\mathbf{GL}(\mathscr{H})$ the multiplicative group of non-singular elements of $\mathscr{L}\mathscr{H}$. Prove that every finite subgroup $\mathscr{G} \subset \mathbf{GL}(\mathscr{H})$ is conjugate to a subgroup of $U(\mathscr{H})$ (the group of unitary operators on $\mathscr{H}$). In other words: There exist some invertible $\mathbf{h} \in \mathbf{GL}(\mathscr{H})$ such that $\mathbf{h}^{-1}\mathbf{g}\mathbf{h}$ is unitary for every $\mathbf{g} \in \mathscr{G}$.
Hint: The operator $Q = \sum_{\mathbf{g}\in\mathscr{G}} \mathbf{g}^*\mathbf{g}$ is positive, and can be used to define a new inner product

$$\langle v,u\rangle_Q = \langle Qv,u\rangle = \sum_{\mathbf{g}\in\mathscr{G}} \langle \mathbf{g}v, \mathbf{g}u\rangle \tag{0.2}$$

and the corresponding norm

$$\|v\|_Q^2 = \sum_{\mathbf{g}\in\mathscr{G}} \langle \mathbf{g}v, \mathbf{g}v\rangle = \sum_{\mathbf{g}\in\mathscr{G}} \|\mathbf{g}v\|^2.$$

Since $\{\mathbf{g} : \mathbf{g} \in \mathscr{G}\} = \{\mathbf{gh} : \mathbf{g} \in \mathscr{G}\}$, we have

$$\langle \mathbf{h}v, \mathbf{h}u\rangle_Q = \sum_{\mathbf{g}\in\mathscr{G}} \langle \mathbf{gh}v, \mathbf{gh}u\rangle = \langle Qv,u\rangle, \tag{0.3}$$

and $\|\mathbf{h}v\|_Q = \|v\|_Q$. Thus, $\mathscr{G}$ is a subgroup of the "unitary group" corresponding to $\langle\cdot,\cdot\rangle_Q$.