Math 114 Midterm (takehome) Exam Spring 2009.

❶ Answer all the problems.

❷ Write clearly! this refers both to the logic of your arguments and to your handwriting.

❸ You may consult books, friends and relations until you feel ready to write the exam. Once you start writing, it is **closed book and no communications.**

1 Prove: If A is a diagonal matrix with distinct entries on the diagonal, and if B is a matrix such that $AB = BA$, then B is diagonal.

2 Let $T \in \mathcal{L}\mathcal{V}$, and $\mathcal{V} = \mathcal{U} \oplus \mathcal{W}$ with both summands T-invariant. Let $\boldsymbol{\pi}$ be the projection onto $\mathcal{U}$ along $\mathcal{W}$.

a. Prove that $\boldsymbol{\pi}$ commutes with T.

b. Is $\boldsymbol{\pi}$ necessarily of the form $P(T)$ for some polynomial P?

c. Does $\boldsymbol{\pi}$ commute with every operator that commutes with T?

3 If $\mathsf{minP}_{\mathrm{T}}$ is irreducible, then $\mathsf{minP}_{\mathrm{T},v} = \mathsf{minP}_{\mathrm{T}}$ for every $v \neq 0$ in $\mathcal{V}$.

4 If $\mathsf{minP}_{\mathrm{T}}$ is irreducible then $\dim \mathcal{V}$ is divisible by $\deg \mathsf{minP}_{\mathrm{T}}$.
Hint: Use Proposition 5.3.2

5 Let $P_1,\ P_2 \in \mathbb{F}[x]$. Prove:

$$\mathsf{ker}(P_1(T)) \cap \mathsf{ker}(P_2(T)) = \mathsf{ker}(\gcd(P_1, P_2)).$$

6 ***(Schur's lemma).*** A system $\{\mathcal{W}, \mathcal{S}\}$, $\mathcal{S} \subset \mathcal{L}(\mathcal{W})$, is *minimal* if no nontrivial subspace of $\mathcal{W}$ is invariant under every $S \in \mathcal{S}$.

Assume that $\{\mathcal{W}, \mathcal{S}\}$ is minimal, and $T \in \mathcal{L}(\mathcal{W})$.

a. If T commute with every $S \in \mathcal{S}$, so does $P(T)$ for every polynomial P.

b. If T commutes with every $S \in \mathcal{S}$, then $\ker(T)$ is either $\{0\}$ or $\mathcal{W}$. That means that T is either invertible or identically zero.

c. With T as above, the minimal polynomial $\mathsf{minP}_{\mathrm{T}}$ is irreducible.

d. If T commute with every $S \in \mathcal{S}$, and the underlying field is $\mathbb{C}$, then $T = \lambda I$.
Hint: The minimal polynomial of T must be irreducible, hence linear.

7 Assume that T is invertible and $\deg \mathsf{minP}_{\mathrm{T}} = m$. Prove that

$$\mathsf{minP}_{\mathrm{T^{-1}}}(x) = cx^m \, \mathsf{minP}_{\mathrm{T}}(x^{-1}),$$

where $c = \mathsf{minP}_{\mathrm{T}}(0)^{-1}$.