Math 114 Final (takehome) Exam Spring 2009.

❶ Answer all the problems.

❷ Write clearly! this refers both to the logic of your arguments and to your handwriting.

❸ You may consult books, friends and relations until you feel ready to write the exam. Once you start writing, it is **closed book and no communications.**

❹ The exam is due by noon, Friday June 5.

1 Consider the $n \times n$ matrix A_n whose entries $a_{i,j}$ are equal to 1 if either $i = 1$ or $j = 1$ (or both), and $a_{i,j} = 0$ otherwise.

a. Find the spectral norm $\|A_n\|_{sp}$, verify that it is a dominant eigenvector, and find the corresponding eigenvector.

b. What are the characteristic and the minimal polynomials of A_n?
Hint: Describe the range of A_n (acting by multiplication on columns in $\mathbb{C}^n$).

c. What of the properties of A_n remain valid for any matrix with positive entries on the first column and on the first row, and zero coefficients everywhere else?

2 Given: $\mathcal{H}$ is a complex n-dimensional inner-product space, $S \in \mathcal{LH}$. Prove that S is normal if, and only if, the orhtogonal complement of every S-invariant subspace is S-invariant.

3 Let $T \in \mathcal{LV}$. Show that there exist vectors $v \in \mathcal{V}$ such that $\mathsf{minP}_{\mathrm{T,v}} = \mathsf{minP}_{\mathrm{T}}$.

4 Let $\mathcal{H}$ be a finite dimensional complex inner-product space, and $\mathbf{GL}(\mathcal{H})$ the multiplicative group of non-singular elements of $\mathcal{LH}$.

Prove that every finite subgroup $\mathcal{G} \subset \mathbf{GL}(\mathcal{H})$ is conjugate to a subgroup of $\mathcal{U}(\mathcal{H})$ (the group of unitary operators on $\mathcal{H}$). In other words: There exist some invertible $\mathbf{h} \in \mathbf{GL}(\mathcal{H})$ such that $\mathbf{h}^{-1}\mathbf{gh}$ is unitary for every $\mathbf{g} \in \mathcal{G}$.
Hint: Show that $\mathcal{G}$ is a subgroup of the "unitary group" corresponding to some appropriate inner-product on $\mathcal{H}$, and explain how this proves the claim.

5 **Nilpotent operators:** An operator T is *nilpotent* if $T^k = 0$ for some integer k, the smallest such k is *the order of* T. The *height of a vector* v (relative to T) is, by definition, the smallest integer l such that $T^l v = 0$.

Assume $T \in \mathcal{L}\mathcal{V}$ is nilpotent.

a. Show that if W is a T-invariant subspace, then the operator $\widetilde{T}$ induced by T on V/W is nilpotent, and the order of $\widetilde{T}$ is not bigger than that of T.

b. Let $v_1 \in V$ be of height k, i.e., $T^k v_1 = 0$, and $T^{k-1} v_1 \neq 0$. Prove that the vectors $\{T^j v_1\}_0^{k-1}$ are linearly independent, and their span V_1 is T-invariant. What is the matrix of $T|_{V_1}$ relative to the basis $u_j = T^{j-1} v_1, \quad j = 1, \dots, k$?

6 Denote by $\mathsf{S_n}$ the group of permutations of $\{1, \dots, n\}$. Let $\{e_j\}_{j=1}^n$ be a basis for $\mathbb{C}^n$.

a. Let $\sigma \in \mathsf{S_n}$, and A_σ the $n \times n$ operator which maps e_i onto $e_{\sigma(i)}$. Describe the spectrum and the eigenvectors of A_σ in terms of the cycle decomposition of σ.

b. Let $a_i > 0$, and let B be the operator defined by $e_i \mapsto a_i e_{i+1}$ for $i < n$, $e_n \mapsto a_n e_1$. What is the spectrum of B?