Intersection Properties of Submanifolds in Euclidean Spaces

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March 10, 2009

Misha and I conjectured that given a Riemannian submanifold $M \subset \mathbb{R}^n$ with positive induced Riemann curvature (i.e., $\Omega(\omega, \omega) > 0$ for all bivectors $\omega$), its intersection with any affine subspace of $\mathbb{R}^n$ is locally a manifold, hence globally also, but possibly with different dimensions on different connected components. It turns out that this is false, as illustrated by the following counterexample.

Consider an $\epsilon$-perturbation $P_\epsilon$ of the $xy$ plane in $\mathbb{R}^3$ into a saddle shape, so that the intersection with the $xy$ plane $P_0$, $P_\epsilon \cap P_0$, is an $X$ shape with the point of intersection at the origin. We view $P_\epsilon$ as a graph over $P_0$ via a map $f_\epsilon$.

Now consider $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, and $P_{\epsilon,t} \subset \{t\} \times \mathbb{R}^3$ embedded in the above manner. Also let $S^2$ be the unit sphere embedded canonically in $\mathbb{R} \times \mathbb{R}^2 \times \{0\}$, i.e., the space $\mathbb{R}^3$ consisting of the union of $P_{\epsilon,t}$, for all time slices $t \in \mathbb{R}$. Furthermore, let $S^2_\epsilon$ be the $\epsilon$ perturbed version of $S^2$, obtained as the union of $f_{\epsilon,t}(S^2_\epsilon \cap \{(t) \times \mathbb{R}^2 \times \{0\})$.

Clearly the intersection $(\mathbb{R} \times P_0) \cap (\mathbb{R} \times P_{\epsilon,t})$ is $\mathbb{R} \times X$, where $X$ is the $X$ shape centered at the origin. The perturbed sphere $S^2_\epsilon$ is certainly still homeomorphic to a round sphere and even have positive curvature everywhere, for small $\epsilon$. Under the map $f_{\epsilon,t}^{-1}$, $\mathbb{R} \times P_\epsilon = \mathbb{R} \times P_0$ which is isometric to $\mathbb{R}^3 \subset \mathbb{R}^4$, and $S^2$ gets mapped to $S^2_\epsilon$ while $\mathbb{R} \times X$ gets mapped to itself, since $f_{\epsilon}$ fixes points on the intersection of $P_\epsilon$ and $P_\epsilon$. Thus we are looking at the intersection of $S^2 \times \mathbb{R} \times X$, which gives an $X$ shape on the 2-sphere, certainly not a manifold.

By the same counterexample, one is led to conclude that no open curvature condition is sufficient to guarantee regular intersection with affine subspaces if the submanifold is not a hypersurface in some affine subspace of the ambient $\mathbb{R}^n$. We further make the following

**Conjecture 0.1.** If $M \subset \mathbb{R}^n$ is a submanifold that is not a hypersurface in any affine subspace of $\mathbb{R}^n$, then there exist some affine subspace $V$ such that $M \cap V$ is not locally a manifold.

which turns out to be false: see the next section.

For hypersurfaces, we do have the following

**Theorem 0.2.** If $M \subset \mathbb{R}^n$ is an $n-1$ dimensional submanifold with positive sectional curvature, then $M \cap V$ is a submanifold locally for any affine linear subspace $V$ of $\mathbb{R}^n$.

**Proof.** It suffices to show that $M$ is locally convex in the sense that for any $p \in M$, the tangent space $T_pM \subset \mathbb{R}^n$ intersects $M$ locally only at $p$. If we express $M$ locally as the orthogonal graph of a map $f : T_pM \to \mathbb{R}$, with $f(p) = p$, then it suffices to have that the curve of intersection of any 2-plane through the unit normal vector $n(p)$ with $M$ locally has curvature of the same sign; i.e., let the curves be parametrized by $\gamma_\alpha(t)$, then $(\frac{d^2}{dt^2}\gamma_\alpha(0), n(p))$ should have the same sign for all $\alpha$. We show that positive sectional curvature guarantees this.

We first claim that if we intersect $M$ by a Euclidean 3-space $E$ through $n(p)$, then the sectional curvature $\Omega_M(T_pM \cap E, T_pM \cap E)$ is precisely the Gauss curvature of the Riemannian manifold $M \cap E$ at $p$.

Let $X,Y$ be two unit length principal directions of $M \cap E$ (which is a locally a manifold at $p$) at $p$. Extend $X$ to a constant velocity vector field along $M \cap \text{Span}\{X, n(p)\}$, say by parametrizing the intersection as a arclength parametrized curve $\gamma_X(t)$, and let $X(\gamma_X(t)) = \frac{d}{dt}\gamma_X(t)$. Similarly extend $Y$ to a constant velocity vector field along $M \cap \text{Span}\{Y, n(p)\}$. Next extend $X$ to a local vector field $\tilde{X}$ on $M \cap E$ near $p$, by translating the curve $\gamma_X$ along $\text{Span}\{Y, n(p)\}$ to span the entire $E$ and letting the extended $\tilde{X}$ be the parallel transport along those translations; since $E$ has a flat Euclidean connection, the transportation is independent of the path chosen. Similarly let $\tilde{Y}$ be the parallel translation of $Y$ along $\gamma_Y$ to all of $E$ along $\text{Span}\{X, n(p)\}$.

Finally with slight abuse of notation again, let $X,Y$ be the vector fields on $M \cap E$ locally near $p$ obtained by orthogonal projection of $\tilde{X}$ and $\tilde{Y}$ onto $TM$. 

Our goal now is to show that
\[ \langle \nabla_X \nabla_Y X - \nabla_Y \nabla_X X - \nabla_{[X,Y]} X, Y \rangle = \left( \frac{d^2}{dt^2} \gamma_X(0), n(p) \right) \langle \frac{d^2}{dt^2} \gamma_Y(0), n(p) \rangle \]

If we choose coordinates on \( M \cap E \) so that \( X, Y \) are the x- and y-axes respectively and \( p \) is the origin, then we have
\[ \langle \nabla_X \nabla_Y X, Y \rangle = X \langle \nabla_Y X, Y \rangle - \langle \nabla_Y X, \nabla_X Y \rangle = X \langle \nabla_X^E X, Y \rangle = \langle \nabla_X^E X, \nabla_Y^E X \rangle \]
where the last equality (2) holds because \( \nabla_X^E X(q) \) is proportional to \( n(p) \) in \( E \), for any \( q \in M \cap E \), by construction of \( X \) as a vector field on \( M \cap E \): for two points on \( M \cap E \) with \( x, y \) coordinates \( (x, y) \) and \((x', y')\), the vector field \( X \) there have the same \( x, y \) coordinates when translated to the origin; 1.5 This implies \( \nabla_X^E \nabla_Y^E X \) is also parallel to \( n(p) \) everywhere, hence orthogonal to \( Y \). 2. \( \nabla_Y^E X = 0 \) because \( \nabla_Y X = 0 \) (see the next paragraph) and \( h(X, Y) = 0 \) where \( h \) is the second fundamental form, by choice of \( X, Y \) as the principal directions.

The second equality (1) because \( \nabla_X Y(p) = \nabla_Y X(p) = 0 \), by definition of induced covariant derivative \( \nabla \) as projection onto \( T(M \cap E) \) of the Euclidean derivative and again the fact that \( \nabla_X^E X(p) \) is proportional to \( n(p) \); 2.
\[ \langle \nabla_Y X, Y \rangle = \langle \text{Proj}_{\text{Span}(X,Y)} \nabla_Y^E X, Y \rangle = \langle \nabla_X^E X, Y \rangle \]
This last point is a very useful trick that will be used again to deal with the second term of the Riemannian sectional curvature:
\[ \langle \nabla_Y \nabla_X X, Y \rangle = Y \langle \nabla_X X, Y \rangle - \langle \nabla_X X, \nabla_Y Y \rangle = Y \langle \nabla_X^E X, Y \rangle = \langle \nabla_X^E X, \nabla_Y^E Y \rangle \]
\[ = \langle \nabla_X^E X, \nabla_Y^E Y \rangle = \left( \frac{d^2}{dt^2} \gamma_X(0), \frac{d^2}{dt^2} \gamma_Y(0) \right) = \langle k_1 n(p), k_2 n(p) \rangle = k_1 k_2 \]
where \( k_1, k_2 \) are the principal curvatures of the surface \( M \cap E \) at \( p \), i.e., the eigenvalues of \( S_p \), the \((1,1)\) type-changed version of \( h \).

The only nonobvious equality above is (3), which amounts to showing \( \langle \nabla_X^E \nabla_X^E X, Y \rangle = 0 \). If we let \( \gamma_{X,y} \) be the integral curves of the vector field \( X \) with \( Y \)-coordinate \( y \), and \( \gamma_{X,y}(0) \) all have \( X \)-coordinate \( 0 \), i.e., abreast with \( p \), then \( \gamma_{X,y}(t) \) all have the same \( X \) coordinate, hence the second derivative \( \frac{d^2}{dt^2} \gamma_{X,y}(t) \) only differ by the \( Z \)-coordinate, i.e., in the direction of \( n(p) \), for all \( t \). Thus \( \nabla_X^E \nabla_X^E X \) is proportional to \( n(p) \), hence orthogonal to \( Y \) at \( p \).

Finally we also have \[ \langle X, Y \rangle = \nabla_X Y - \nabla_Y X = 0 \] by torsion-freedom. So indeed the sectional curvature of \( M \) along \( T_p M \cap E \) is equal to the scalar curvature in the usual surface theory in \( \mathbb{R}^3 \).

Next we show that near \( p \), \( M \) always lies on one side of \( T_p M \). This amounts to showing all normal sectional curves of \( M \) at \( p \) have the same curvature sign, and keeps the same sign for some fixed finite time interval around \( 0 \): we can define a function \( k : S^{n-2} \rightarrow \mathbb{R} \) by the line curvature at \( p \) of the normal sectional curve through \( p \) with direction given by an element in \( S^{n-2} \subset T_p M \). Now suppose for the sake of contradiction that no matter how small \( \epsilon \) is, there is some \( v \in S^{n-2} \) such that the normal section along \( v \) through \( p \) changes curvature sign within \((-\epsilon, \epsilon)\). Then we can choose a sequence of \( v_i \) whose limit in \( S^{n-2} \), which we call \( v_\infty \) leads to a normal section which cannot have positive line curvature at \( p \). This effectively shows there is a neighborhood of \( p \) restricted to which normal sections all have line curvatures of the same sign, i.e., they lie on one side of \( T_p M \), hence \( M \) is locally convex.
If $V$ is an affine subspace of $\mathbb{R}^n$ and $V$ intersects $M$ at $p$, then it either is tangent to $M$ in which case locally it only intersects $M$ at $p$ alone, a 0-manifold, or $V$ intersects $T_p M$ along a subspace of dimension greater than 0, which means the intersection is transverse at $p$, hence also is a manifold locally.

Thus if $M \subset V \subset \mathbb{R}^n$ is a hypersurface with positive sectional curvatures in $V$ which in turn is affine linear in $\mathbb{R}^n$, then for any other $W$ affine linear in $\mathbb{R}^n$, $M \cap W = M \cap (V \cap W)$ which is a submanifold of $V$, by the above theorem, hence a submanifold of $\mathbb{R}^n$.

1 Regular affine intersection

We shall say a Riemannian manifold $(M, g)$ has regular affine intersection property (RAIP) if there is an imbedding in $\mathbb{R}^n$ such that $M$ intersecting any affine subspace in $\mathbb{R}^n$ gives a local submanifold.

What we have shown before was that manifolds that can be imbedded as positively curved hypersurfaces have RAIP. The converse is certainly not true, as shown by non-planar curves in $\mathbb{R}^3$.

**Proposition 1.1.** As long as the curve $C$ is analytic, meaning its coordinates are analytic functions, its intersection with any affine subspace $V$ locally must be either isolated points or connected components.

**Proof.** Suppose a point $p = C(t_0)$ in the intersection $C \cap V$ is not isolated. Then the tangent vector of $C$ at $p$ must lie in $V$. Similarly all its higher covariant derivatives $C^{(n)}(t_0)$ must lie in $V$, by looking at higher difference quotients, whose limiting values agree with the corresponding derivatives by looking at the Taylor expansion. This shows that the composition $g \circ C(t)$, where $g$ is the distance function from $V$, has all derivatives being zero at $t_0$, hence must be identically zero near $t_0$ by analyticity. Thus there is an open neighborhood of $p$ in $C$ that lies in $V$, i.e., $C \cap V$ is locally diffeomorphic to $\mathbb{R}$ at $p$. We can now look at connected component $K$ of $p$ in $C \cap V$, and at each point in $K$, $C \cap V$ must be locally diffeomorphic to $\mathbb{R}$ (otherwise the point is isolated). So $K$ is a 1-dimensional submanifold of $C$, hence must be diffeomorphic to either an open interval or a circle. The second case clearly is a connected component. If $K$ is an open interval, and if its end points are not in $C$, then we also get connected component. If $K$ has an end point $q = C(t_1)$ in $C$, i.e., $q \in C \setminus (K \setminus K)$, then since all derivatives of $C$ at $q$ from one side are zero, by continuity of derivatives, the derivatives from the other side must also be zero. Hence by analyticity again $g \circ C(t)$ must be identically zero near $t_1$. □

Higher dimensional analogue of the above result is not true, since the classification of 1-dimensional manifold played an important role in the proof.

Our next order of business is to study some standard examples that fall out of the range above. The first coming to mind is $S^1 \times S^1$ with the standard flat metric imbedded in $\mathbb{R}^4$.

We don’t need to consider its intersection with $\mathbb{R}^4$. So we start with codimension 1. Let $V$ be affine codimensional 1 in $\mathbb{R}^4$. Let $v$ be one of its unit normal vectors. Then intersection with $V$ is the same as taking the preimage under the function $g : \mathbb{R}^4 \to \mathbb{R}$, $g(x) = \langle v, x \rangle$ of some point $c \in \mathbb{R}$, where $c$ is the distance from $V$ to the origin of $\mathbb{R}^4$.

Now if we restrict $g$ to $S^3 \times S^1$, we can apply Morse theory results to conclude that the nondegenerate critical points of $g$ are isolated. The preimage of regular values are certainly smooth submanifolds by regular value theorem of differential topology (see [3]). So we just need to show that all critical points are nondegenerate.

Let $v = (v_1, v_2, v_3, v_4)$.

Also parametrize $S^1 \times S^1$ by $\phi : \mathbb{R}^2 \to \mathbb{R}^4$, where

$$\phi(\theta_1, \theta_2) = (\cos \theta_1, \sin \theta_1, \cos \theta_2, \sin \theta_2)$$

Then we have

$$g \circ \phi(\theta_1, \theta_2) = v_1 \cos \theta_1 + v_2 \sin \theta_1 + v_3 \cos \theta_2 + v_4 \sin \theta_2$$

so $\phi(\theta_1, \theta_2)$ is a critical point if

$$-v_1 \sin \theta_1 + v_2 \cos \theta_1 = 0 \quad (5)$$
$$-v_3 \sin \theta_2 + v_4 \cos \theta_2 = 0 \quad (6)$$

and is furthermore degenerate if the determinant of the Hessian is 0:

$$\begin{pmatrix}
-v_1 \cos \theta_1 + v_2 \sin \theta_1 & 0 \\
0 & -(v_3 \cos \theta_2 + v_4 \sin \theta_2)
\end{pmatrix}$$
this implies either $v_1 \cos \theta_1 + v_2 \sin \theta_1$ or $v_3 \cos \theta_2 + v_4 \sin \theta_2$ is zero. Say without loss $v_1 \cos \theta_1 + v_2 \sin \theta_1 = 0$. Then with (5) this implies $v_1 = v_2 = 0$. This further implies $S_1 \times S_1 \cap V = \phi(\{(\theta_1, \theta_2) : v_3 \cos \theta_2 + v_4 \sin \theta_2 = c\})$. This is a disjoint union of circles in $\mathbb{R}^4$ since the solutions in $\theta_2$ to $v_3 \cos \theta_2 + v_4 \sin \theta_2 = c$ are isolated and finite by solving a quadratic equation.

So $C \cap V$ is a finite union of isolated points and disjoint compact 1-manifolds (diffeomorphic but not necessarily isometric to circles).

By the real analytic implicit function theorem (see [2]), we also know that each of the compact 1-manifold in the intersection is actually real analytic manifolds, since the regular value theorem is proved by applying implicit function theorem to $f : M \to N$, with $f$ being a real analytic submersion, realized as $\tilde{f} : \mathbb{R}^{m-n} \times \mathbb{R}^n \to \mathbb{R}^n$ in some local chart, in which $\tilde{f}$ has nonsingular derivative with respect to the last $n$ variables.

Next we consider $W$ affine and codimension 2. We can take an arbitrary $V$ affine and codimension 1, containing $W$. Since $V \cap C$ is locally either an isolated point or a compact 1-manifold, we just have to deal with the case when $W$ is an affine 2-plane in $\mathbb{R}^3$ and $C := V \cap S_1 \times S_1$ is a closed analytic curve in $\mathbb{R}^3$.

Observe that $C$ actually lies in a round 2-sphere in $W$, since $S^1 \times S^1 \subset S^3(\sqrt{2})$ in $\mathbb{R}^4$ and intersection of a sphere with an affine subspace is also a sphere of possibly lower dimension. Thus we are really looking at the intersection of two curves on $S^2$. But by analyticity of $C$, we know that $C \cap V$ must be a submanifold by the lemma above.

Finally we come to one dimensional affine subspace, say $L$. Then we are really looking at the intersection of a line in $\mathbb{R}^3$ with a curve on $S^2$, which is clearly a collection of at most two points.

So we have shown that $(S^1 \times S^1, \text{flat})$ satisfies RAIP.

References

