COLORING OF MEET-SEMILATTICES

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Abstract. Given a commutative semigroup $S$ with 0, where 0 is the unique singleton ideal, we associate a simple graph $\Gamma(S)$, whose vertices are labeled with the nonzero elements in $S$. Two vertices in $\Gamma(S)$ are adjacent if and only if the corresponding elements multiply to 0. The inverse problem, i.e., given an arbitrary simple graph, whether or not it can be associated to some commutative semigroup, has proved to be a difficult one. In this paper, we extend results by DeMeyer[3], McKenzie, and Schneider[4] on this problem by studying the complement of graphs. As an application and an extension of work in [3] we prove that every compact connected 2-manifold admits an Eulerian triangulation that can be associated to a zero divisor semigroup graph.

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1. Introduction

In their joint paper [4], F.R. DeMeyer, T. McKenzie, and K. Schneider first defined commutative zero-divisor semigroup graphs, and verified in the language of category theory that the functorial relation between semigroups and graphs is covariant. Given a commutative semigroup \( S \) with 0, the associated zero-divisor graph \( \Gamma(S) \) is a graph whose vertices form one-to-one correspondence with the nonzero elements in \( S \), and two vertices are joined by an edge if the product of their corresponding elements in \( S \) is zero. Recently, Frank DeMeyer and Lisa DeMeyer formulated four necessary conditions for a graph to be a commutative zero-divisor semigroup graph. Based on explicit calculation, they concluded that those four conditions are actually sufficient for all commutative zero-divisor semigroup graphs up to five vertices [3]. They also obtained positive results on certain families such as refinement of star graphs and graphs that have semi-lattice structure. Progress has stopped, however, for graphs with more than five vertices. A six-vertex counterexample given in [3] (see the next section) shows that the four conditions are not sufficient for larger graphs. The proof that it is not a zero-divisor semigroup graph is quite tortuous.

The present paper does not continue with the classification of zero-divisor semigroup graphs based on the number of vertices. Instead, for each graph, we look at its complement graph, whose edges are precisely the nonedges of the original graph. For example, the complement of a complete graph on \( n \) vertices is simply the totally disconnected graph on \( n \) vertices. For any zero-divisor semigroup \( S \), we denote by \( \Gamma(S) \) the graph whose vertices are in 1-1 correspondence with elements in \( S \) and two vertices are joined by an edge if and only if their product is nonzero. The task of deciding whether a given graph is a zero-divisor semigroup graph is then reduced to the following problem:

Given a graph \( G \), does there exist a zero-divisor semigroup \( S \) such that \( G = \Gamma(S) \). \( G \) is said to be admissible if such \( S \) exists. At first this is nothing more than a restatement of the original problem. But surprisingly, it provides much more information about the associability of \( G \) and enables us to classify large families of commutative zero-divisor semigroup graphs. We will henceforth deal primarily with the complement of graphs, rather than the original graphs as studied in the previous literature [3], [4].

The main results of this paper can be summarized as following:

1. Given a graph \( G \) with connected components \( G_\alpha \), \( G \) is admissible if and only if each \( G_\alpha \) is, provided that none of the \( G_\alpha \) is a singleton. As a special case, this result generalizes the semi-lattice result in [3].

2. We show that every graph (not necessarily connected) is the induced subgraph of a connected admissible graph.

3. We reformulate the four admissibility conditions in [3] by defining a new invariant called depth. Several other admissibility criteria are given with respect to certain condition on depth.

4. We define the notion of graph separability, which captures certain intrinsic properties of graphs in terms of their subgraphs (more accurately, subcollection of vertices). The end result will be a coloring criterion to determine a graph to be nonadmissible, which in turn can be used to prove many familiar families of graphs with symmetries to be nonadmissible.
5. The final section of this paper addresses a question posed in [3] and partially answered in [8]: given any regular surface (or 2-manifold), can we always find a triangulation that is the graph of some commutative zero-divisor semigroup? We settle this question in the affirmative by inductively constructing such triangulations.

2. Resume of previous results and motivations for new ones

We will abbreviate commutative zero-divisor semigroup simply by semigroup. Because of the functorial correspondence between the graph and the semigroup, we will speak loosely of the product of two vertices to mean the product of the two elements represented by the two vertices. Given a graph $G$ and any two vertices $a, b \in V(G)$, we denote by $d(a, b) \in \mathbb{Z}_+ \cup \{\infty\}$ the length of the shortest path between $a$ and $b$. If $S$ is a subgraph of $G$, and $a, b \in V(S)$, we further denote by $d_S(a, b)$ the length of the shortest path in $S$ between $a$ and $b$.

Many of the general results in this paper were motivated by theorems and examples given in [3]. We list some of them here.

Recall the neighborhood of a vertex $x$ in a graph $G$, denoted $N(x)$ is the set of vertices adjacent to $x$ and we further denote the closure $\overline{N(x)} = N(x) \cup \{x\}$.

The authors of [3] showed that a simple graph $G$ must satisfy the following four conditions in order to be the graph of some commutative zero-divisor semigroup:

1. $G$ is connected.
2. The distance between any two vertices is $\leq 3$.
3. Any induced cycle in $G$ is either a 3-cycle or a 4-cycle.
4. For every pair $x, y \in V(G)$ not adjacent, $\exists z \in V(G)$ such that $N(x) \cup N(y) \subset \overline{N(z)}$.

These four conditions give complete characterization of zero-divisor semigroup graphs with up to five vertices. One major limitation is that the proof of their necessity only uses information about the adjacency of the graph. The square of any vertex is not considered. Since the graph is simple, in particular does not allow loops, we do not have information about self-adjacency.

In Theorem 3 of [3], two types of graphs, complete bipartite graphs and refinement of star graphs, are shown to be zero-divisor semigroup graphs. Refinement simply means some of the off-center vertices may be adjacent to each other. The result on complete bipartite graphs easily generalizes to complete $n$-partite graphs.

One example that shows the insufficiency of the four conditions on graphs with more than 5 vertices is the 6 vertices graph $G$ below (see example 2 of [3]):

The proof that there does not exist any zero-divisor semigroup $S$ such that $\Gamma(S) = G$ is fairly complicated. The main idea is case analysis, and clever use of symmetry in $G$.

The complement of $G$, $\overline{G}$ looks like the following (with vertices labeled the same):

Notice that if the central triangle $\Delta a_1a_2a_3$ were removed, the induced graph on the remainder of the vertices form a disconnected graph in which each connected component was originally attached to exactly one vertex in the central triangle. In this case, we say $\overline{G}$ is separable by its subgraph $Z = \{a_1, a_2, a_3\}$. In fact, $\overline{G}$ is strongly separable by $Z$, which means that each vertex in $Z$ was originally connected to a nonempty component of the resulting induced graph after $Z$ is removed. In general, if a graph is strongly separable by one of its subgraphs, then we can show that it is not admissible, provided that it satisfies some additional
minor conditions. The inadmissibility of $\mathcal{G}$ then follows easily from the general consideration of separable graphs.

3. DISJOINT UNION OF GRAPHS

Definition 1. A simple graph $G$ is said to be the disjoint union of $\{G_i\}_{i \in A}$ if $\forall i \in A$, $G_i$ is a connected simple graph, $V(G) = \bigcup_{i \in A} H_i$, $\forall a \in H_i$, $b \in H_j$, $i \neq j$, $(a, b) \notin E(G)$, and the induced subgraph of $V(H_i)$ in $G$ is isomorphic to $G_i$ itself for each $i \in A$.

Theorem 1. Let $G$ be the disjoint union of connected graphs $\{G_i\}_{i \in A}$:
1) If $|V(G_i)| = 1$ for some $i$, then $G$ is always admissible.
2) If $|V(G_i)| \geq 2 \forall i \in A$, then $G$ is admissible if and only if each $G_i$ is.

Proof. (1) Suppose $V(G_i) = \{x\}$ for some $i \in A$. Then the complement of $G$ is the refinement of a star graph. Thus, by Theorem 2 in [3], $G$ is admissible. This proves (1).

(2) Assume $|V(G_i)| \geq 2$ for all $i \in A$. Assume the graphs $G_i$ are admissible for all $i \in A$ and denote a semigroup associated to the graph $G_i$ by $S(G_i)$. Let $S(G)$ be the semigroup whose elements are the vertices in $G$ where the semigroups $S(G_i)$ are all subsemigroups of $G$. If $a \in V(G_i)$ and $b \in V(G_j)$ where $i \neq j$, define $ab = 0$. Clearly $S(G)$ satisfies the adjacency rules of $G$ and $S(G)$ is associative since any triple product is either 0 or all three elements must lie in the same give subsemigroup $S(G_i)$.

To prove the converse of (2) suppose $G$ is admissible and $S(G)$ is an associated semigroup (not necessarily unique). It suffices to show that each set $V(G_i) \cup \{0\}$ must be closed under multiplication of $G$. Let $(a, b) \in E(G_i)$ where $E$ denotes the edge set, and suppose that $ab = c \in V(G_j)$ for some $j \neq i$. By assumption $|V(G_j)| \geq 2$ and $G_j$ is connected. Hence $N(c) \neq \emptyset$. Let $d \in N(c)$. Then $cd \neq 0$, but $cd = (ab)d = a(bd) = 0$, a contradiction. Therefore the set $V(G_i)$ forms a subsemigroup of $S(G)$. Hence $G_i$ is admissible.

Example 1. The graph below is admissible since it has a singleton component $G_1 = \{a\}$. But the right component is a non-admissible example (see [3]):

4. EXTENSION TO ADMISSIBLE GRAPHS

Observe that from the main result of last section, any graph can be imbedded as the induced graph of an admissible graph. The easiest way to do that is by adding an additional vertex disjoint from all vertices of the original graph. The following results show that it is even possible to imbed any graph as the subgraph of a \emph{connected} admissible graph.

Theorem 2. Given a connected graph $G$, let $G'$ be the graph obtained by the following procedure:
For each edge $(a, b)$ in $G$, add a vertex $c_{a,b}$ and edges $(a, c_{a,b})$, $(b, c_{a,b})$ to $G$.

Then $G'$ is connected and admissible.

Proof. Since $G$ is connected and each additional vertex $c_{a,b}$ is joined with a vertex in $G$, we see that $G'$ is also connected.

To see it’s admissible, we define its multiplication table as follows:
for each edge $(a, b)$ in $G$, define all multiplication between two elements from the set $\{a, b, c_{a,b}\}$ to be $c_{a,b}$.
To check for associativity, the only nontrivial case is when \((a,b)\) and \((b,d)\) are both edges of \(G\). In that case \((ab)d = c_{a,b}d = 0\) because \(c_{a,b}\) is disjoint from \(d\). Similarly \(a(bd) = 0\). Also it is straightforward to see that the multiplication between \(\{a,b,c_{a,b}\}\) is associative.

\[\square\]

**Remark.** Note that the above construction can be slightly generalized as follows: instead of adding one vertex for each edge \((a,b)\) in \(G\), one may add more than one vertices \(c_{a,b}^{1},c_{a,b}^{2},\ldots,c_{a,b}^{n}\), and form the complete graph among the \(n+2\) vertices \(a,b,c_{a,b}^{1},\ldots,c_{a,b}^{n}\). Note that \(n \geq 1\) can be chosen differently for different edges. By the same argument as before, one can show that the resulting connected graph is admissible.

**Corollary.** Let \(G\) be an arbitrary graph not necessarily connected. Then there always exists a connected admissible graph \(H'\) of which \(G\) is an induced subgraph.

**Proof.** If \(G\) is connected, then \(G'\) from theorem 2 above satisfies the requirement for \(H'\): the only thing left to check is that \(G\) is an induced subgraph of \(G'\). But since there are no edges added in \(G'\) between vertices in \(G\), the induced subgraph on the original vertices in \(G\) coincide with the graph \(G\).

Now suppose \(G\) has more than one connected components \(\{G_{i}\}_{i \in A}\) where \(A\) is some index set. Assume \(A\) is well ordered with the smallest element 0 and let \(x_{i} \in V(G_{i})\) be an arbitrary vertex in \(G_{i}\) for each \(i\). Now construct a graph \(H\) by introducing a vertex \(y_{i}\) for each \(0 \neq i \in A\), and join \(y_{i}\) to \(x_{i}\) and \(x_{i-1}\) with an edge, where \(i-1\) is the element in \(A\) immediately preceding \(i\). Since there is a path between any two connected components, \(H\) is connected.

Finally let \(H'\) be constructed from \(H\) as in theorem 2. Then \(H'\) is a connected admissible graph. Furthermore \(G\) is the induced subgraph on the vertex set \(V(H) - \{y_{i}\}_{i \in A}\), as is easily checked.

\[\square\]

**Example 2.** To illustrate the above construction, let \(G\) be the following graph: which is nonadmissible by the four criteria in [3] followed by theorem 1.

Now let \(H'\) be the following graphs:

Then \(H'\) is admissible by theorem 2 and \(G\) is an induced subgraph of \(H'\). Notice that for the left component of \(G\), we have extended using the 3-clique construction exactly as described in theorem 2, whereas for the right component, we have used the generalized \(n\)-clique construction described in the remark following the theorem.

5. **Boundary and Depth Associated With Subgraphs**

Given a simple graph \(G\) and two vertices \(a,b \in V(G)\), recall that \(d(a,b)\) denotes the length of the shortest path from \(a\) to \(b\) and \(d\) is in fact a metric on \(G\); in particular it satisfies the triangle inequality.

**Definition 2.** Let \(G\) be a simple connected graph and let \(S \subseteq V(G)\), then \(x\) is said to be **bound** \(S\) if for every \(y \in N(x)\), \(\max\{d(y,t) | t \in S\} \leq 1\). The set of boundary vertices of \(S\) in \(G\) is denoted by \(B_{G}(S)\).

\(S \subseteq V(G)\) is said to be bounded if \(B_{G}(S) \neq \phi\); otherwise \(S\) is said to be unbounded.

The most common subset \(S\) in the above definition will be either a singleton, or an edge. See remark after the corollary to Theorem 3.
Example 3. Let $G$ be the following graph:

Observe that the vertex set $\{A, B\}$ is bounded by $E$, since the only vertices adjacent to $E$ is $F$, which is adjacent to both $A$ and $B$. Check also that no other vertex bounds $\{A, B\}$: for example $A$ does not bound $\{A, B\}$ because $D$ is adjacent to $A$, but $D$ is not adjacent to $B \in \{A, B\}$. Thus in the notation above, $B_G(\{A, B\}) = \{E\}$.

On the other hand, the edge $\{C, D\}$ is unbounded. Take $A$ as a candidate for its boundary: $B$ is adjacent to $A$ yet $B$ is not in the neighborhood of $D$. Hence $A \not\in B_G(\{C, D\})$. In the same manner one can show that all the other vertices are not in $B_G(\{C, D\})$. Thus $B_G(\{C, D\}) = \emptyset$.

To motivate Theorem 3 below, assume $G$ is admissible. If we define $CD = A$, then $(CD)B = AB \neq 0$, but $C(DB) = C0 = 0$ a contradiction. Similarly if we define $CD$ to be any other vertex in $G$, the same type of contradiction will result. On the other hand defining $AB = E$ does not give contradictions of this sort.

Theorem 3. (Boundary Condition) If $G$ is admissible, then $\forall a, b \in V(G)$ not necessarily distinct, $ab \in B_G(\{a, b\}) \cup \{0\}$. In particular, every edge in $G$ must be bounded.

Proof. Suppose $(a, b)$ is an edge in $G$. Then $ab \neq 0$. Let $c = ab$ and suppose that $c \notin B_G(\{a, b\})$. By definition of boundary, there exists a vertex $y \in N(c)$ so that $d(a, y) > 1$ or $d(b, y) > 1$. Hence either $ya = 0$ or $yb = 0$ so that $0 = yab = yc$. This is a contradiction.

Note that if $(a, b)$ is not an edge in $G$ then $ab = 0$.

In practical computation, one only needs to check vertices "sufficiently close" to the vertex set of interest in order to determine whether it is bounded or not. For instance, any vertex whose distance with any vertex in the set is bigger than 2 can be automatically excluded from consideration.

Corollary. Every vertex in an admissible graph $G$ must be bounded.

Proof. Simply observe that $B_G(X) \subset B_G(Y)$ if $X \subset Y \subset V(G)$.

Example 4. Using the boundary condition, one sees that any cycle $C_n$ of length $n \geq 5$ is not admissible: take any edge $(a, b)$ in $C_n$, we show that $B_{C_n}(\{a, b\}) = \emptyset$ (see graph below). First of all $a \notin B_{C_n}(\{a, b\})$, because if $c$ is adjacent to $a$ and $c \neq b$, then $c$ is adjacent to a fourth vertex $x \neq a$. But $x$ is not adjacent to $a$ otherwise $b - a - c - x - b$ will be a 4-cycle. Similarly $b \notin B_{C_n}(\{a, b\})$. Now suppose $c$ is adjacent to $b$, $c \neq a$, and $c \in B_{C_n}(\{a, b\})$. Let $x$ be adjacent to $c$ but $x \neq b$. Then $x$ is adjacent to neither $a$ nor $b$. So again $c$ cannot be in the boundary of $(a, b)$. So we are left with the possibility of a boundary vertex $x$ with $d(x, a) \geq 2$ and $d(x, b) \geq 2$. But since $n \geq 5$, this implies that either $d(x, a) \geq 3$ or $d(x, b) \geq 3$, which is impossible by definition of boundary.

In a completely analogous manner, one can prove that a path $P_n$ of length $n \geq 6$ is not admissible.

Remark. It is not the case that every clique in an admissible graph must be bounded. Note that the clique $\{A, B, C\}$ in the graph below is unbounded, but the graph itself is admissible by Theorem 2.
In a straightforward manner, one can prove the following restatement of Theorem 1 in [3], using the boundary condition proved in this section:

**Theorem 4.** If $G$ is admissible then the following four conditions hold:
1. $G$ can have at most 1 nontrivial component, i.e., with more than 1 vertices.
2. For any connected pair $a, b \in V(G)$, $d(a, b) \leq 3$.
3. the induced cycles in $G$ are either 3- or 4- cycles.
4. For every nonadjacent pair $a, b \in V(G)$, $\mathcal{N}(a) \cup \mathcal{N}(b) \subset \overline{\mathcal{N}(c)}$ for some $c \in V(G)$.

In later sections, we will also need a notion closely related to boundary, which will serve as a convenient marker for case analysis.

**Definition 3.** Let $S$ be a subset of the vertices in $G$. The *depth of $S$ in $G$*, denoted $D_G(S)$, is defined as follows: if $S$ consists of a single vertex of degree $\leq 1$, then $D_G(S) = 0$; otherwise $D_G(S) = \min\{d(t, b)|t \in S, b \in B_G(S), t \neq b\}$.

We will be mainly interested in the depth of a single vertex or an edge, just as in the case of boundary set.

**Example 5.** For a complete graph $K_n$ on $n > 1$ vertices, every nonempty subgraph $S$ has depth 1, since for every vertex in $K_n$, each of its neighboring vertices is either adjacent to all vertices in $S$, or is one of the vertices in $S$ and adjacent to the rest. In either case, since $n > 1$, the minimum nonzero distance between boundary vertices and vertices in $S$ is 1.

Some properties of depth are summarized in the following propositions.

**Proposition 1.** Let $(a, b) \in E(G)$. Then $B_G(\{a, b\}) \subseteq B_G(a) \cap B_G(b)$.

Proof. Suppose $x \in B_G(\{a, b\})$, then forall $y \in \mathcal{N}(x)$, $d(y, a) \leq 1$ and $d(y, b) \leq 1$, hence $y \in B_G(a)$ and $y \in B_G(b)$ \implies $y \in B_G(a) \cap B_G(b)$. \hfill \Box

**Proposition 2.** For a connected graph $G$ with at least two vertices, $D_G(S) = 0$ if and only if $S$ consists of a single vertex of degree 1.

Proof. One direction is just the definition. If $S$ does not consist of a single vertex of degree 1, then the second part of the definition of depth always yields a positive integer. \hfill \Box

**Proposition 3.** Let $S \subset V(G)$ and suppose $a \in B_G(S)$, then $d(a, s) \leq 2$, $\forall s \in S$.

Proof. Suppose $d(a, s) > 2$ for some $s \in S$, let $t \in \mathcal{N}(s)$. Then $d(a, t) > 1$ by triangle inequality, which shows that $a \not\in B_G(S)$. \hfill \Box

The results below provide a quick way to see that a graph is not admissible.

**Proposition 4.** For bounded $a, b \in V(G)$, $|D_G(a) - D_G(b)| \leq d(a, b)$.

Proof. If $a = b$, then equality holds. If $d(a, b) = 1$, then the only way $|D_G(a) - D_G(b)| > d(a, b)$ is if, without loss of generality, $D_G(a) = 2$ and $D_G(b) = 0$. This can happen only if $\deg(b) = 1$, which implies $\mathcal{N}(b) = \{a\}$. Since $d(a, a) = 0$, $b \in B_G(a)$. Hence $D_G(a) \leq 1$, a contradiction. Finally if $d(a, b) \geq 2$, then $|D_G(a) - D_G(b)| \leq d(a, b)$ always holds because depth is bounded above by 2, below by 0. \hfill \Box

As a special case, we have
Corollary. For adjacent \( a, b \in V(G) \), \( |D_G(a) - D_G(b)| \leq 1 \).

If \( S \) consists of a single vertex \( a \), then the above proposition says the distance from any boundary vertex of \( S \) to \( a \) must be less than 2. We also have the following

Corollary. The depth of any subset \( S \) in \( G \) cannot exceed 2.

Lemma 1. Suppose \( G \) is admissible and \( (a, b) \in E(G) \). Then \( d(ab, a) \leq 2 \), and similarly for \( b \).

Proof. By theorem 3 (boundary condition), \( ab \in B_G(\{a, b\}) \cup \{0\} \). Since \((a, b)\) is an edge in \( G \), \( ab \neq 0 \). Hence \( ab \in B_G(\{a, b\}) \). It follows from the definition of boundary that \( d(ab, a) \leq 2 \) and \( d(ab, b) \leq 2 \).

\( \square \)

6. VERTEX POWERTIES

The boundary condition only uses the adjacency data of the graph. In particular, it does not take into account the square of individual vertices. One technique to prove that a graph is not admissible, therefore, is to assume it is admissible first and then find contradictions by assigning certain values to the square of each vertex.

Let \( S \) be a zero divisor semigroup and let \( G = \Gamma(S) \) be its associated graph, i.e., the complement of \( G \) is the zero divisor semigroup graph associated with \( S \) in the sense of [3]. Let \( v \) be a vertex of \( G \) which also represents an element in \( S \).

\( v \) is said to be 2-nilpotent if \( v^2 = 0 \);

as usual \( v \) is idempotent if \( v^2 = v \).

If \( v \) is not 2-nilpotent or idempotent, then \( v^2 \) is equal to some nonzero element in \( S \) not equal to itself. then \( v \) is said to be transpotent. In fact the distance \( d(v, v^2) \) can be at most 2 by the boundary condition. Therefore we have two possibilities: \( v \) is said to be 1-transpotent (resp. 2-transpotent) if \( d(v, v^2) = 1 \) (resp. \( d(v, v^2) = 2 \)).

Note that the meaning of the prefix number is different from that in "2-nilpotent".

Thus every element in a semigroup graph can only have one of the four possible potencies: idempotent, 2-nilpotent, 1-transpotent, and 2-transpotent.

Proposition 5. Let \( G \) be an admissible graph. For any \( a \in V(G) \), \( a^2 \in B_G(a) \cup \{0\} \).

Proof. Suppose \( a \) is not 2-nilpotent, then for each \( b \) in \( N(a^2) \), \( ba^2 \neq 0 \). Therefore \( ba \neq 0 \), which implies \( b \in \overline{N(a)} \). Hence by definition of boundary, \( a^2 \in B_G(a) \). \( \square \)

Proposition 6. Suppose \( G \) is admissible. If two adjacent vertices \( a, b \in V(G) \) are both 2-nilpotent, then they must be adjacent to a common vertex.

Proof. We prove a stronger version of the depth claim. In fact we have \( d(ab, a) = 2 \) and \( d(ab, b) = 2 \). Suppose without loss of generality \( d(ab, a) < 2 \), then \( d(ab, a) < 2 \) by lemma 1, and we have two cases to consider: if \( d(ab, a) = 1 \), then \( (ab)a \neq 0 \), but \( (ab)a = a^2b = ab = 0 \), a contradiction. If \( d(ab, a) = 0 \), then \( d(ab, b) = 1 \) and we have a symmetric argument.

Now for any \( y \in \overline{N(ab)} \), \( d(y, a) \leq 1 \) and \( d(y, b) \leq 1 \). By the triangle inequality, \( d(y, a) + d(y, ab) \geq d(a, ab) \) and similarly for \( b \), so we are forced to have \( d(y, a) = 1 \), \( d(y, b) = 1 \). Thus \( y \) is a vertex adjacent to both \( a \) and \( b \), as desired. \( \square \)

Proposition 7. Let \( G \) be admissible. If \( (a, b) \in E(G) \) and \( a \) is 2-nilpotent, then \( ab \notin N(a) \).
Proof. Suppose $d(ab,a) = 1$. Then since $ab \neq 0$, $(ab)a \neq 0$. But $(ab)a = a^2b = 0$, a contradiction.

Proposition 8. Let $G$ be admissible. If $(a,b) \in E(G)$ and $a$ is idempotent, then $ab \in N(a) \cup \{a\}$.

Proof. Suppose not, then $(ab)a = 0$. But $(ab)a = a^2b = ab \neq 0$.

Corollary. Let $G$ be admissible. If $(a,b) \in E(G)$, a 2-nilpotent and $b$ idempotent, then $ab \in N(b) - N(a)$.

Proof. This follows immediately from the previous two propositions.

Lemma 2. If $v \in V(G)$ is not 2-nilpotent, then $v^2 \in B_G(v)$.

Proof. Suppose $v^2 = u \not\in B_G(v)$. Then there exists $x \in N(u)$, $d(x,v) \geq 2$. So $xv^2 = (xv)v = 0$. But since $u$ is adjacent to $x$, $xv^2 = xu \neq 0$, a contradiction.

7. GRAPH SEPARABILITY

Motivated by the six-vertex non-admissible graph presented in the introduction, we define the notion of graph separability.

Let $Z$ be a subgraph of $G$, with $x \in Z$. Let $B$ be the induced subgraph of $G$ on the vertex set $(V(G) \setminus V(Z)) \cup \{x\}$. Let $T_x$ denote the connected component of $B$ containing $x$. The induced subgraph on $V(T_x) \setminus \{x\}$ is called the foliage of $x$ with respect to $Z$, denoted by $F_{olZ}(x)$. Once fixed, $Z$ will be called the center of $G$. The vertex $x$ will be called the anchor of $F_{olZ}(x)$ or that of $T_x$.

Definition 4. A graph $G$ is said to be separable by a subgraph $Z$ if the following two conditions hold:
1. $V(G) \setminus V(Z) = \bigcup_{x \in Z} V(F_{olZ}(x))$;
2. For $u \neq v \in Z$, $F_{olZ}(u) \cap F_{olZ}(v) = \emptyset$.

Definition 5. Let $G$ be separable by $Z$. $G$ is said to be strongly separable by $Z$ if each vertex of $Z$ has nonempty foliage.

Example 6. Consider the following graph $G$:
Let $Z = \{A, B, C\}$. Then $G$ is strongly separable by $Z$. In particular, $F_{olZ}(A) = \{D\}$, $F_{olZ}(B) = \{E\}$, $F_{olZ}(C) = \{F\}$, none of which is empty.
On the other hand, if we let $Z = \{A, B, C, D\}$, then $G$ is still separable by $Z$, but no longer strongly separable because the foliages of $A$ and $D$ are empty.
Finally if we let $Z = \{A, B, D\}$, then $G$ is not even separable by $Z$.

We will be mainly interested in the strongly separable case. Further results on weakly separable graphs will be given in a forthcoming paper.

Theorem 5. If $G$ is separable by a subgraph $Z$, then $G$ is connected if and only if $Z$ is connected.

Proof. By definition of separability, $Z$ connected implies $G$ connected.

Conversely, suppose $Z$ is not connected, then $Z = C_1 \cup C_2$, where $C_1$ and $C_2$ are disjoint, nonempty, and no vertex in $C_1$ is adjacent to any vertex in $C_2$. Let $x \in C_1$ and $y \in C_2$, and let $P = \{v_0, v_1, \ldots, v_k\}$ be a path from $x$ to $y$, where $v_0 = x$ and
\(v_k = y\). Let \(\alpha = \max\{i | v_i \in C_1\}\) and \(\beta = \max\{i | v_i \not\in C_2\}\). Then \(\alpha < \beta\) because \(\{i | v_i \in C_1\} \subset \{i | v_i \not\in C_2\}\) and \(\alpha \neq \beta\) since otherwise \(v_\alpha\) and \(v_\beta+1\) will be adjacent, contradicting the fact that \(C_1\) and \(C_2\) are disjoint. But then the set of vertices \(\{v_\alpha+1, v_\alpha+2, \ldots, v_\beta\}\) forms a nonempty subpath of \(P\) that’s not contained in \(C_1\) or \(C_2\), hence \(\{v_\alpha+1, v_\alpha+2, \ldots, v_\beta\} \not\subseteq C_1 \cup C_2 = Z\) and it is connected to \(v_\alpha \in C_1\) and \(v_\beta+1 \in C_2\), hence by definition, \(\{v_\alpha+1, v_\alpha+2, \ldots, v_\beta\} \subseteq Fol_Z(v_\alpha) \cap Fol_Z(v_\beta+1)\). So \(G\) is not separable by \(Z\). 

\[\square\]

**Lemma 3.** Let \(G\) be an admissible graph strongly separable by a connected subgraph \(Z\) with at least two vertices, and let \(x \in Z, y \in Fol_Z(x)\). Then \(xy \in Fol_Z(w)\), with \(w \neq x\) in \(Z\). Then \(xy\) must be adjacent to \(w\) otherwise its distance from \(x\) or \(y\) will exceed 2, which contradicts Proposition 3. Thus \(xyw \neq 0\). But \(y\) is not adjacent to \(w\) (see figure below).

Now suppose \(xy = w\), and let \(u \in Fol_Z(w)\) guaranteed by strongly separability. Then \(xyu \neq 0\) yet \(ux = 0 = uy\).

Finally suppose \(xy = x\). Then \(xyw \neq 0\) for \(w \in Z\) adjacent to \(x\). But \(y\) is not adjacent to \(w\).

So we must have \(xy \in Fol_Z(x)\), which recall does not include the vertex \(x\) itself. 

\[\square\]

**Theorem 6.** Let \(G\) be admissible and strongly separable by connected \(Z\) with \(|V(Z)| \geq 2\), with \(x, y \in Z\) adjacent. Then \(xy \in ((V(Fol_Z(x)) \cap N(x)) \cup (V(Fol_Z(y)) \cap N(y))) \cap L(G)\), where \(L(G)\) denotes the set of degree 1 vertices in \(G\). Thus either \(Fol_Z(x)\) or \(Fol_Z(y)\) contains a singleton component.

**Proof.** Suppose \(xy = w \in Z\), and \(u \in Fol_Z(w)\), then \((xy)u \neq 0\). But \((xu)y = 0\) since we may assume without loss that \(x \neq w\). This shows \(xy \in V(G) \setminus V(Z)\).

Next suppose \(xy = u \in Fol_Z(w)\) for \(w \in Z\) possibly equal to \(x\) or \(y\) (See figure below). If \(u\) is adjacent to some other \(v \in Fol_Z(w)\), then \(xuv \neq 0\). But \(xu = 0\) implies \(xyv = 0\). This shows \(xy \in L(G)\).

If \(w \neq x, y\), then \(u\) must be adjacent to \(w\) by boundary consideration. Thus \(w(xy) = uw \in Fol_Z(w)\) by the above lemma. Also \(x\) must be adjacent to \(w\) because \(wxy \neq 0\) and \(wx\) is adjacent to \(y\). Since \(wx \notin Z\), we know that \(wx\) must be in the foliage of \(y\). Therefore \((xv)u \in Fol_Z(y)\) by the same lemma. But foliages of distinct anchor vertices are disjoint by definition of strong separability. So we conclude that \(w \in \{x, y\}\).

Combining the above argument, \(xy\) must be in either \(Fol_Z(x)\) or \(Fol_Z(y)\) and adjacent to the anchor vertex. Furthermore it must not be adjacent to any other vertex in the foliage.

\[\square\]

Given a strong separable admissible graph, one can often find admissible induced subgraphs from its foliages.

**Theorem 7.** Let \(G\) be admissible and strongly separable by a connected \(Z\) with more than 1 vertex and let \(x \in Z\).

1. \(Fol_Z(x) \cup \{0\}\) and \(T_x \cup \{0\}\) are both closed under multiplication of \(G\).

2. Let \(U\) be the union of the singleton components of \(Fol_Z(x)\). For each component \(V \subset Fol_Z(x)\), \(V \cup U \cup \{0\}\) and \(V \cup U \cup \{x, 0\}\) are closed under the multiplication of \(G\).
Proof. 1. Let \( a, b \in \text{Fol}_Z(x) \) be adjacent. Suppose \( ab \notin \text{Fol}_Z(x) \). Then \( ab \) is adjacent to some \( c \notin T_x \). But \( ac = 0 \) since \( a \) and \( c \) are not adjacent.

If \( x^2 = w \) for some \( w \neq x \) in \( Z \), let \( y \in \text{Fol}_Z(w) \cap N(w) \), then \( x^2y \neq 0 \), contradicting that \( x \) and \( y \) are not adjacent.

If \( x^2 = u \in \text{Fol}_Z(w) \), \( w \neq x \), then clearly \( u \in N(w) \). By Lemma 3, \( uw \in \text{Fol}_Z(w) \). But then \( x^2w \neq 0 \) implies \( xw \in \text{Fol}_Z(x) \). An easy application of the boundary condition then shows \( (xw)x \neq u \).

Combining the above argument with Lemma 3, we get 1.

2. Let \( x, y \in V \) and suppose \( xy = u \in W \), a different component of \( \text{Fol}_Z(x) \) with at least 2 vertices. Then \( u \) is adjacent to some other vertex in \( W \), to which \( x \) and \( y \) are not adjacent. This shows \( V \cup U \cup \{0\} \) is closed. By a similar argument, \( V \cup U \cup \{0, x\} \) is also closed under the multiplication of \( G \).

\[ \square \]

The final theorem in this section gives a useful coloring criterion to prove that a given graph is not admissible.

**Theorem 8.** Let \( G \) be strongly separable by a connected subgraph \( Z \), and let \( x, y \in Z \) be adjacent. If \( G \) is admissible then none of the following can happen:

1. \( x \) and \( y \) are both 2-nilpotent;
2. \( x \) and \( y \) are both idempotent;
3. \( x \) and \( y \) are both transient.

**Proof.** Let \( G \) be admissible, we will arrive at a contradiction for each of the three cases.

1. By theorem 6, we may without loss of generality assume \( xy = a \in \text{Fol}_Z(x) \), where \( a \) is adjacent to \( x \). So \( x(xy) \neq 0 \), contradicting that \( x \) is 2-nilpotent.

2. Again let \( xy = a \in \text{Fol}_Z(x) \). Then \( (xy)y = 0 \) but \( xy^2 = xy \neq 0 \).

3. Let \( xy = a \in \text{Fol}_Z(x) \). Then \( ax \neq 0 \). But \( yx^2 = 0 \) since \( x^2 \in \text{Fol}_Z(x) \), not adjacent to \( y \). \[ \square \]

**Corollary.** Let \( G \) is strongly separable by a connected induced subgraph \( Z \):

1. Suppose for each \( x \in Z \), \( \text{Fol}_Z(x) \) has a component contained in \( N(x) \). Then \( G \) is not admissible provided that \( Z \) cannot be 2-colored. In particular, the six-vertex graph in Example 6 is not admissible.

2. If \( Z \) contains a (not necessarily induced) complete subgraph on \( n \) vertices where \( n \geq 4 \), then \( G \) is not admissible.

**Proof.** 1. Applying Theorem 7 (2) and 8, we see each vertex in \( Z \) must be either idempotent or transient and adjacent ones must be of different potencies in order for \( G \) to be admissible. For the second statement, observe that the six-vertex graph given there is strongly separable by the central 3-clique, with singleton foliages. And a 3-clique is not 2-colorable.

2. Let \( W \) be the complete subgraph of \( Z \). Then it is impossible to 3-color \( W \). So by Theorem 8, \( G \) is not admissible. \[ \square \]

Under the conditions of part 1 (resp. part 2) in the above corollary, we can further deduce that if \( Z \) contains a subgraph (not necessarily induced) that cannot be 2-colored (resp. 3-colored), then \( G \) is not admissible. In particular, if \( G \) satisfies the conditions of part 1 and \( Z \) contains a cycle of odd length, then \( G \) is not admissible.
Remark. There is an obvious generalization of separability, namely when some anchor vertices in Z have overlapping foliages. All results in this section except Theorem 8 carry over in that case, provided that the restricted distance \( d_{T_i,T_j}(x,y) \) between any pair of anchor vertices \( x \) and \( y \) with common foliage is far enough. We have not computed the critical distance to ensure validity of the above results, which may well depend on the number of anchors that share a single foliage. Nevertheless, it is easy to see that 5 is an upper bound.

Finally we list some technical conditions that can be used to show certain strongly separable graph to be nonadmissible, or help in constructing explicit multiplication table for a separable graph.

**Proposition 9.** Let \( G \) be admissible and strongly separable by a connected \( Z \) with more than 1 vertex and let \( x \in Z \).

1. If \( d(y,x) \geq 2 \) for all \( y \in B_G(x) \cap \text{Fol}_Z(x) \), then \( x \) must be 2-nilpotent.
2. If \( \text{Fol}_Z(x) \) has a component \( V \) contained in \( N(x) \), then \( x \) is not 2-nilpotent.
3. If \( \text{Fol}_Z(x) \) has no singleton component and has a component \( V \) such that every vertex in \( B_G(x) \cap V \) is at least distance 2 away from \( x \), then \( x \) is not idempotent.
4. If \( \text{Fol}_Z(x) \) has components contained in \( N(x) \), then \( x \) is not 2-transpotent.

**Remark.** In the case there is no singleton component in \( \text{Fol}_Z(x) \), 1 implies that each component is closed.

**Proof.** 1. Suppose not, then \( x \) is either idempotent or 2-transpotent. In the idempotent case, let \( y \in \text{Fol}_Z(x) \cap N(x) \): \( y(x^2) = yx \neq 0 \) but \( (xy)x = 0 \) since \( d(xy,x) \geq 2 \) by the boundary condition. The transpotent case is similar.

2. Let \( y \in V \). Then \( xy \in N(x) \), otherwise \( d(xy,y) \geq 3 \), which is not possible by boundary consideration. Now suppose \( x^2 = 0 \). We immediately get a contradiction: \( (xy)x \neq 0 \).

3. Let \( a \in N(x) \cap V \). By 3, \( ax \in V \) and therefore \( d(ax,x) = 2 \). Thus \( (ax)x = 0 \), a contradiction.

4. Let \( y \) be a vertex in a component contained in \( N(x) \). Then \( xy \) is also in a component contained in \( N(x) \). Thus \( x^2y \neq 0 \). Now suppose \( d(x,x^2) = 2 \). Then \( xy \) is not in the same component of \( \text{Fol}_Z(x) \) as \( x^2 \) by 2. Hence \( (xy)x = 0 \), a contradiction.

\( \Box \)

8. CONSTRUCTING SEMIGROUPS ON TRIANGULATION GRAPHS OF COMPACT SURFACES

The association of zero-divisor semigroups to triangulation graphs was first studied in [3] as a possible topological invariant of compact manifolds. Partial results have been obtained in [8], where it was shown that the Klein bottle and the real projective plane do admit triangulation whose 1-skeleton is a zero-divisor semigroup graph (in the non-complement sense). While it is difficult to understand dimension 3 or higher, the result below completely characterizes associability in the case of surfaces.

Recall [7] that a triangulation of a compact surface \( S \) is a finite family \( T \) of triangles \( T_i, 1 \leq i \leq n \) such that

1. \( \bigcup_{i=1}^{n} T_i = S \);
2. If \( T_i \cap T_j \neq \emptyset \), then \( T_i \cap T_j \) is either a common edge of \( T_i \) and \( T_j \) or a common vertex of \( T_i \) and \( T_j \).
Theorem 9. Every compact 2-manifold admits a triangulation whose 1-skeleton is a zero-divisor semigroup graph.

Proof. We use the standard polygonal model of compact surfaces. Thus a surface of Euler characteristic $n$ may be obtained by gluing a regular $2(2-n)$-gon, denoted by $a_1 - a_2 - \ldots - a_m$, where $m = 2(2-n)$ and $a_i$'s are vertices arranged in a fixed order. Also denote by $A_i$ the edge between $a_i$ and $a_{i+1 \mod m}$. As an exception, the sphere is represented by a 2-gon with the opposite sides glued in opposite orientation.

Now we triangulate the polygon in the following way:
1. Place a smaller copy of the polygon concentrically inside the original polygon. Label its vertices $b_1$ through $b_m$ so that $a_i$, $b_i$, and the center of the polygon are colinear.
2. Trisect each $A_i$ and label the resulting two additional vertices $a'_i$, $a''_i$, so that $a_i$, $a'_i$, $a''_i$, $a_{i+1 \mod m}$ form a path.
3. Join $b_i$ separately to $a_i$, $a'_i$, and $a''_i$ with an edge.
4. Fix some $b_i$. Join each $b_j$, $|j-i| > 1 \mod m$, to $b_i$ with an edge.

An octagon triangulated with the above scheme is illustrated in the figure below. It is easy to check that this is a valid triangulation.

We claim that the resulting graph $G$ is a zero-divisor semigroup graph. First note that $a_i$ all represent the same vertex in the triangulation, which we denote by $a$. Similarly the primed and doubly primed vertices are identified in pairs, with two different schemes according as the surface is orientable or not. In either case, the complement of $G$ contains a singleton component $\{a\}$, because $a$ is adjacent to every other vertex in the 1-skeleton by construction. Hence $G$ is a zero-divisor semigroup graph by Theorem 1.

\[\square\]

References


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