A generalization of the union-closed set conjecture

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The union closed set conjecture states the following: if $B$ is a finite set, and $A \subseteq \mathcal{P}(B)$ is a collection of subsets closed under the union operation. Then there is at least one element $b \in B$ that appears in at least half of all the sets in $A$.

We propose the following natural generalization of this conjecture

**Conjecture 0.1.** Let $|B| = n$ be the size of $B$. For any $k \leq n$ positive integer, there exists at least one subset $S \subseteq B$ of size $k$ such that it is contained in at least $2^{-k}|A|$ of the sets in $A$.

Notice that when $k = 1$ we have the Union closed set conjecture.

Below we prove the conjecture for the cases $k = n, n-1, n-2$.

**Proof.** We translate the problem in the language of binary string. So each element of $A$ is represented as a binary number of $n$ bits, if we fix an ordering of the elements of $B$. And union corresponds to bit-wise operation.

$k = n$. This requires only that $B$ is contained in one of the sets in $A$, only when $|A| \geq 2^n$, but this means $|A| = 2^n$ and it's the power set of $B$, which in particular contains $B$.

$k = n-1$. We divide into three cases based on the size of $A$. When $|A| < 2^{n-1}$, we need 0 sets that contains a subset of size $n-1$. Well that means we don’t need to do anything.

If $|A| = 2^n$, then $A$ is the powerset of $B$, which certainly contains 2 sets each of which contains some particular subset of size $n-1$, for example: $(1, \ldots, 1, 0), B$ would work.

If $|A| \geq 2^{n-1}$, we need to show there is some $S$ with $|S| = n-1$ such that it’s contained in at least one set of $A$. Let $T$ be the total number of digits spanned by elements in $A$. This means that if any element in $B$ is contained in some set in $A$, that element will be recorded in $T$. If $T = n-1$, then $A$ must be the power set of some subset of $B$ of size $n-1$, in which case we can take $S$ to be the biggest element of $A$. Otherwise $T = B$, which means $B \in A$, and certainly any subset of $B$ of size $n-1$ can be our $S$.

Finally we deal with $k = n-2$.

We can easily eliminate the $|A| < 2^{n-2}$ and $A = \mathcal{P}(B)$ case as before. So there are three cases left:

$|A| \geq 2^{n-2}$. Here we can use the same argument for $k = n-1$ and $|A| \geq 2^{n-1}$ case before to show there is some $S \subseteq B$ with $|S| = n-2$ such that $S$ is contained in at least one element of $A$.

$|A| \geq 2^{n-1}$. If the digit span $T < n$, then we will have the powerset of some subset of $B$ of size $n-1$, and in that case we basically reduce to the $|A| = 2^n$ case of $k = n-1$. So we may assume $T = n$, then whatever $S$ we choose in the end, it’s gonna be contained in the largest element of $A$, namely $B$.

So we just need to show that besides $B$, there is at least one other set in $A$ that has size larger than or equal to $n-2$. Suppose within $A \setminus \{B\}$ the set $U$ with largest number of elements has $n-j$ elements, then we can represent it without loss of generality as $(1, \ldots, 1, 0, \ldots, 0)$, with $n-j$ 1’s and $j$ 0’s. Any other set $V$ that contains any of the other elements must contain all of them, since otherwise the union of $U \cup V$ will violate the maximality assumption about $U$. This eliminates $(2^j - 2)2^{n-j-1}$ of the set of binary sequences of size $n$, since the last $j$ digits can only be of the form $(0, \ldots, 0)$ or $(1, \ldots, 1)$. Since $j$ is at least 3 if we assume there is no element of size $n-2$ in $A \setminus \{B\}$, we are left with at most $2^{n-2}$ elements. But $|A|$ is at least $2^{n-1} > 2^{n-2}$, so $U$ must have more than $n-3$ elements. But then $U$ is the other set besides $B$ that contains a subset of size $n-2$. Here we see one important argument which shows the existence of some set of size at least $n-j$ based on the number of sets in $A$; we call this the minimal length argument.

$|A| \geq 3(2^{n-2})$. Again $T n$. By the minimum length argument, we get that there must exist a set $U$ in $A \setminus \{B\}$ of size $n-1$. Let’s denote it by $(1, \ldots, 1, 0)$. Consider all sets of the form $(x, \ldots, x, 1)$. There are at least $2^{n-2}$ of them, with equality if and only if all sets of the form $(x, \ldots, x, 0)$ are in $A$. We want to show that among such sets there is at least one with length $n-1$, for then that set will share a
subset of length $n - 2$ with $U$, which together will share that subset with $B$ and give us three sets in $A$ containing a subset of length $n - 2$.

We restrict to the first $n - 1$ coordinates $(x, \ldots, x)$, and suppose that the maximum length is $n - 1 - j$, achieved by $V$. Without loss of generality write $V = (1, \ldots, 1, 0, \ldots, 0)$ with $n - 1 - j$ 1’s and $j$ 0’s. Then by union-closedness of $V$ (since the last entry 1 is attracting), we can eliminate all sequences with the last $j$ digits not equal to either $(0, \ldots, 0)$ or $(1, \ldots, 1)$. That’s $(2^j - 2)2^{n-j-1}$ sequences and we are left with $2^{n-j}$ entries. Since there are at least $2^{n-2}$ sequences with last coordinate 1 in $A$, $j \leq 2$. But in the case $j = 2$, all subsets of $V$ union the last element, i.e., all strings of the form $(x, \ldots, x, 0, 0, 1)$ are in $A$. And all strings of the form $(x, \ldots, x, 0)$ are in $A$ as well from the last paragraph. So for example $(1, \ldots, 1, 0, 0), (1, \ldots, 1, 0) \in A$. These together with $(1, \ldots, 1)$ give the three sets containing the first $n - 2$ elements of $B$. Otherwise $j = 1$ and $(1, \ldots, 1, 0, 1) \in A$. This together with $(1, \ldots, 1, 0)$ and $(1, \ldots, 1)$ also finishes the proof.

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