RESEARCH STATEMENT

JENYA SAPIR

My main areas of interest are geometric topology and geometric group theory. More specifically, I’m interested in ways in which objects associated to hyperbolic surfaces, like the curve complex and Teichmuller space, exhibit properties of hyperbolic spaces. I am also interested in ways to use dynamics to study surfaces and surface spaces such as Teichmuller space, for example using geodesic flows. Relatedly, I am interested in the behavior of geodesics in these spaces.

My current research project is to better understand geodesics on a hyperbolic surface $S$. I am getting coarse, but explicit bounds on the number of closed geodesics on $S$ given upper bounds on length and self-intersection number. I am also extending a classic result of Birman and Series. Their result is that the set of points on $S$ that lie on complete geodesics with at most $K$ self intersections is nowhere dense and has Hausdorff dimension 1 for any $K$. I am extending it to the case of complete geodesics with infinitely many self-intersections that are in some sense sparsely distributed with respect to arclength. Underlying both of these results is a combinatorial model for both closed geodesics and geodesic arcs on $S$ that sends each geodesic $\gamma$ to a word $w(\gamma)$ in a finite alphabet. The word $w(\gamma)$ can be used to coarsely recover geometric properties of $\gamma$ like length and self-intersection number.

1. Counting Result

There was a lot of work done in the 70’s and 80’s on counting closed geodesics on a negatively curved surface $S$ from the point of view of counting closed orbits of the geodesic flow. Let $\mathcal{G}^c$ be the set of closed geodesics on $S$ and let

$$\mathcal{G}^c(L) = \{ \gamma \in \mathcal{G}^c | l(\gamma) \leq L \}$$

where $l(\gamma)$ is the length of $\gamma$. The famous result in Margulis’s thesis states that if $S$ is negatively curved with a complete, finite volume metric, then

$$(1.0.1) \quad \# \mathcal{G}^c(L) \sim \frac{e^{\delta L}}{\delta L}$$

where $\delta$ is the topological entropy of the geodesic flow, and where $f(L) \sim g(L)$ if $\lim_{L \to \infty} \frac{f(L)}{g(L)} = 1$ [Mar70]. (Note that $\delta = 1$ when $S$ is hyperbolic.)

Recently, there has been work on the dependence of the number of closed geodesics on their self-intersection number. If $i(\gamma, \gamma)$ is the transverse self-intersection number of $\gamma \in \mathcal{G}^c$, let

$$\mathcal{G}^c(L, K) = \{ \gamma \in \mathcal{G}^c | l(\gamma) \leq L, i(\gamma, \gamma) \leq K \}$$

Then we can pose the following question.

- If $K = K(L)$ is a function of $L$, what is the asymptotic growth of $\# \mathcal{G}^c(L, K)$ in terms of $L$?
As part of her thesis, Mirzakhani showed that for a hyperbolic surface $S$ of genus $g$ and with $n$ punctures,
\[ \#G^c(L, 0) \sim c(S) L^{6g-6+2n} \]
where $c(S)$ is a constant depending only on the geometry of $S$ [Mir08]. For geodesics with one self-intersection, Rivin has shown that
\[ \#G^c(L, 1) \sim c'(S) L^{6g-6+2n} \]
where $c'(S)$ is another constant depending only on the geometry of $S$ [Riv12].

For arbitrary functions $K = K(L)$, no asymptotic bounds are known. However, we can make progress on the following question.

- Given arbitrary $L$ and $K$, what are the best upper and lower bounds we can get on $\#G^c(L, K)$?

A trivial upper bound comes from the fact that $\#G^c(L, K) \leq \#G^c(L)$, but this bound does not have any dependence on $K$. By combining the asymptotic result in [ABEM12] with results in [Bas13, Ker80, Wol79], we can get that for large $L$ and arbitrary $K$,
\[ \#G^c(L, K) \approx f(K) L^{6g-6} \]
where $f(K)$ is the (finite) number of mapping class group orbits in $G^c(L, K)$. We write $A \asymp B$ if there is a constant $c$ depending only on the geometry of $S$ s.t. $\frac{1}{c} B \leq A \leq cB$.

I am working on better understanding the dependence of $\#G^c(L, K)$ on $K$. I am getting explicit, but coarse, bounds on the size of $G^c(L, K)$ for an arbitrary hyperbolic surface $S$. As will be explained later, we can reduce the problem of bounding $\#G^c(L, K)$ on an arbitrary hyperbolic surface to the problem of bounding $\#G^c(L, K)$ on a hyperbolic pair of pants. The following result for closed geodesics on a pair of pants will appear in my thesis.

**Theorem 1.1.** On a hyperbolic pair of pants $P$, we get the following upper bound
\[ \#G^c(L, K) \leq \min\{e^{c\sqrt{K} \log \left(\frac{1}{\sqrt{K}}\right)}, e^{c\sqrt{K} \log (c\sqrt{K})}\} \]
where $c$ is a constant depending on the geometry of $P$.

To make sense of this theorem, it is useful to think of the upper bounds $L$ and $K$ competing with each other for control over geodesics. Furthermore, given a random geodesic $\gamma_L \in G^c(L)$ for each $L$, $i(\gamma_L, \gamma_L) \sim \kappa L^2$ almost always as $L$ tends to infinity, where $\kappa$ depends only on the geometry of the surface (in this case, on the geometry of $P$) [Lal96]. Thus, one should think that $K \approx L^2$ in the theorem above.

When $K = L^2$, the theorem gives us roughly that $\#G^c(L, K) \approx e^{cL}$, which is a coarse version of (1.0.1). Note that although Margulis proved (1.0.1) for closed surfaces, this result can be extended to many surfaces with boundary, such as pairs of pants, using Patterson-Sullivan measures [Pat76, Sul79]. For example, these measures are used to show that (1.0.1) holds for a compact hyperbolic surface with totally geodesic boundary in the case where $\delta > 1/2$ in [Gui86]. For a different approach that shows (1.0.1) without the restriction on $\delta$, see [Lal89].

One consequence of this theorem is that there are exponentially fewer geodesics in $G^c(L)$ with $i(\gamma, \gamma) << L^2$:
Corollary 1.2. On a pair of pants $\mathcal{P}$, if $K = K(L)$ is a function of $L$ such that $K = o(L^2)$, then
\[
\frac{\#\mathcal{G}^c(L, K)}{\#\mathcal{G}^c(L)} < e^{-cL}
\]
for all $L$, and for $c$ depending only on the geometry of $\mathcal{P}$.

Note that there are finitely many closed geodesics on $\mathcal{P}$ with at most $K$ self-intersections. This is a generalization of the fact that a pair of pants has only three simple closed geodesics. This is in contrast with any other surface, where there are infinitely many.

2. Birman Series Type Result

On a closed hyperbolic surface $S$, the geodesic flow is mixing, which implies that the set of all closed geodesics is dense (see, eg, [Bow08, Section 3]). This means that we can get as close as we like to any point on $S$ with a closed geodesic. It is interesting to study closed geodesics that do not exhibit this behavior.

Birman and Series showed that complete (not necessarily closed) geodesics on a surface $S$ with finitely many self-intersections are actually quite restricted in where they can go on a surface. Let $\mathcal{G}$ be the set of complete geodesics on $S$ parameterized by arc length. Then let
\[
S_K = \{ \gamma \in \mathcal{G} \mid i(\gamma, \gamma) \leq K \}
\]
and let $\mathcal{T}_K$ be the set of points that lie on a geodesic in $S_K$. (Since almost all complete geodesics have infinitely many self-intersections, the ones in $S_K$ should be thought of as almost simple.)

Theorem 2.1 (Birman-Series [BS85]). For each $K$, the set $\mathcal{T}_K$ is nowhere dense and has Hausdorff dimension 1.

They prove this theorem for all hyperbolic surfaces $S$, and in fact, it holds more generally for all negatively curved surfaces.

This result is used, for instance, to show the famous McShane identity for curves on a surface with cusps [McS98]. The identity on a once-punctured torus, found in his thesis [McS91] is as follows
\[
\sum_{\gamma \in \mathcal{S}_0} \frac{1}{e^{l(\gamma)} - 1} = \frac{1}{2}
\]
and the one for a general surface with cusps is similar. There are many extensions of this identity to, for instance, closed surfaces and surfaces with boundary, for example [LT11].

Roughly, the Birman-Series result says that while we can get as close as we like to any point on $S$ with some closed geodesic, very few points can be approached by geodesics in $S_K$. As previously mentioned, complete geodesics with finitely many self-intersections are rare. Given $\gamma : \mathbb{R} \to S \in \mathcal{G}$, let $\gamma_l = \gamma|_{]-1,1[}$ be a subarc of length $l$ centered at $\gamma(0)$. Then $i(\gamma_l, \gamma_l) \sim kl^2$ almost always (see, for example, [Lal, Section 2.3]). (Note that the geodesic flow is not ergodic with respect to Lebesgue measure on a surface with boundary, but we can use Patterson-Sullivan measures instead.) Thus, not only do most complete geodesics have infinitely many self-intersections, but they also occur at a rate of roughly $l^2$ self-intersections per length $l$ subarc. So we can try to answer the following question.
For what sets $G' \subset G$ of complete geodesics with infinitely many self-intersections does the set $F'$ of points lying on some geodesic in $G'$ satisfy the conclusion in [BS85]? That is, for what $G'$ is $F'$ nowhere dense and of Hausdorff dimension 1?

Once again, the case of a pair of pants is a good starting point to answer this question. In the case of a hyperbolic pair of pants $P$, we extend the Birman-Series theorem to cover, in some sense, a maximal set of rare complete geodesics. For each $\varepsilon > 0$, let

$$G_\varepsilon = \{ \gamma \in G \mid \limsup_{l \to \infty} \frac{i(\gamma_l, \gamma_l)}{l^2} < \varepsilon \}$$

So these are geodesics with infinitely many self-intersections, where the self-intersection points occur slightly less frequently than on an average complete geodesic. Let $F$ and $F_\varepsilon$ be the sets of points lying on some geodesic in $G$ and $G_\varepsilon$, respectively.

On a pair of pants, complete geodesics no longer cover the whole surface. However, if $\delta \in (0, 1)$ is the topological entropy of the geodesic flow on $P$, then the Hausdorff dimension of $F$ is $1 + 2\delta$ for $\delta < 1/2$, and $F$ has points of Lebesgue density when $\delta > 1/2$ (e.g., see [HJHL12].) We get that for $\varepsilon$ small enough, the Hausdorff dimension of $F_\varepsilon$ is smaller than the Hausdorff dimension of all of $F$. And in fact, the Hausdorff dimension of $F_0$ is 1, where $F_0$ is the set of points lying on some complete geodesic $\gamma$ with $i(\gamma_l, \gamma_l) = o(l^2)$.

**Theorem 2.2.** The Hausdorff dimension of $F_\varepsilon$ is at most $\mu(\varepsilon) + 1$, where $\mu(\varepsilon)$ is a function so that $\lim_{\varepsilon \to 0} \mu(\varepsilon) = 0$. In particular, the Hausdorff dimension of $F_0$ is 1.

The Birman-Series theorem also says that their set is nowhere dense. Here, our result differs from theirs:

**Proposition 2.3.** If $\bar{F}_\varepsilon$ denotes the closure of $F_\varepsilon$, then

$$\bar{F}_\varepsilon = F$$

for every $\varepsilon \geq 0$.

So if $\delta > 1/2$, where $\delta$ is the topological entropy of the geodesic flow, $F_\varepsilon$ cannot be nowhere dense.

In fact, the proof of this proposition implies that even if we take $S_{<\infty} = \{ \gamma \in G \mid i(\gamma, \gamma) < \infty \}$, and if $T_{<\infty}$ is the set of points lying on geodesics in $S_{<\infty}$, then $T_{<\infty} = F$.

The problem is that $G_\varepsilon$ is too big to get a nowhere density result. We need more regularity for how quickly $i(\gamma_l, \gamma_l)$ goes below $\varepsilon l^2$ as $l$ goes to infinity. So for each positive $L$, let

$$G_\varepsilon(L) = \{ \gamma \in G_\varepsilon \mid i(\gamma_l, \gamma_l) < 5\varepsilon l^2, \forall l \geq L \}$$

This is a subset of $G_\varepsilon$, so in particular, the Hausdorff dimension holds for this set. Let $F_\varepsilon(L)$ be the set of all points lying on some geodesic in $G_\varepsilon(L)$.

**Theorem 2.4.** There is an $\varepsilon_0$ so that for all $\varepsilon < \varepsilon_0$, $F_\varepsilon(L)$ is nowhere dense for all $L$.

It is also of interest to look at the local picture. For most points $p$ on a surface $S$, and for most directions at $p$ (chosen with respect to a measure that puts full measure on directions corresponding to complete geodesics), the geodesic rays in those directions cover most of $F$ and their self-intersections behave in the same way.
as most complete geodesics. We want to understand the behavior of the directions at a point \( p \) that do not behave in the usual way.

For any point \( p \) on a pair of pants \( \mathcal{P} \), we look at vectors at \( p \) that point in the direction of complete geodesic rays with sparse self-intersections. So let \( \mathcal{R} \) be the set of complete geodesic rays at a point \( p \), parameterized by arc length. If \( \rho \in \mathcal{R} \), let \( \rho_l = \rho|_{[0,l]} \). Let

\[ \mathcal{R}_\epsilon = \{ \rho \in \mathcal{R} \mid \limsup_{l \to \infty} \frac{i(\rho_l, \rho_l)}{l^2} < \epsilon \} \]

and let

\[ \mathcal{R}_\epsilon(L) = \{ \rho \in \mathcal{R}_\epsilon \mid i(\rho_l, \rho_l) < 5\epsilon l^2, \forall l > L \} \]

Let \( \mathcal{V}_\epsilon \) and \( \mathcal{V}_\epsilon(L) \) be the set of tangent vectors of rays in \( \mathcal{R}_\epsilon \) and \( \mathcal{R}_\epsilon(L) \), respectively. Then we the following versions of the above theorems for the case of directions at \( p \).

**Theorem 2.5.** The Hausdorff dimension of \( \mathcal{V}_\epsilon \) is at most \( \mu'(\epsilon) \), where \( \mu'(\epsilon) \) is a function so that \( \lim_{\epsilon \to 0} \mu'(\epsilon) = 0 \). In particular, \( \mathcal{V}_0 \) has Hausdorff dimension 0.

**Theorem 2.6.** There is an \( \epsilon_0 \) so that for all \( \epsilon < \epsilon_0 \), \( \mathcal{V}_\epsilon(L) \) is nowhere dense for all \( L \).

The above results on a pair of pants all come from a combinatorial model for closed geodesics and geodesic arcs on \( \mathcal{P} \) that assigns a word \( w(\gamma) \) to each geodesic \( \gamma \). This model is in the same vein as combinatorial model, for example, used to prove the Birman-Series theorem stated above. The difference is that the words \( w(\gamma) \) give finer controls over the self-intersections of \( \gamma \).

### 3. Work in progress

We are working on the details of a lower bound on the size of \( G^c(L, K) \) for a pair of pants:

**Theorem 3.1.** For \( L \) and \( K \) large enough, we get that

\[ 2^{c\sqrt{K}} \leq \# G^c(L, K) \]

where \( c \) depends on the geometry of \( \mathcal{P} \).

We are currently working on versions of Theorem 1.1 and Theorems 2.2 through 2.6 for a general surface \( S \). However, while on a pair of pants we get exponentially fewer closed geodesics in \( G(K, L) \) than in all of \( G^c(L) \) as soon as intersection number and length satisfy \( K = o(L^2) \), on a general surface we expect to start getting exponentially fewer geodesics only when \( K = o(L^{4/3}) \). So we expect the following upper bound.

**Conjecture 1.** On an arbitrary hyperbolic surface \( S \),

\[ \# G^c(L, K) \leq \min \{ e^{-L}, e^{K^{3/4}\log q(K^{3/4}, L)} \} \]

where all constants depend on the geometry of \( S \) and \( q(\cdot, \cdot) \) is a rational function in two variables.

This counting result should allow us to extend the Birman-Series result on an arbitrary surface. However, the order of growth of the number of self-intersections
on the complete geodesics will have to be smaller. On a surface $S$, we will now define
\[ \mathcal{G}_\epsilon = \{ \gamma \in \mathcal{G} \mid \limsup_{l \to \infty} \frac{i(\gamma_l, \gamma_l)}{l^{4/3}} < \epsilon \} \]
and
\[ \mathcal{G}_\epsilon(L) = \{ \gamma \in \mathcal{F}_\epsilon \mid i(\gamma_l, \gamma_l) < 5\epsilon l^{4/3}, \forall l > L \} \]
Let $\mathcal{F}_\epsilon$ and $\mathcal{F}_\epsilon(L)$ be the set of points lying on some geodesic in $\mathcal{G}_\epsilon$ and $\mathcal{G}_\epsilon(L)$, respectively. We expect the following Birman-Series-type result.

**Conjecture 2.** On an arbitrary hyperbolic surface $S$, the set $\mathcal{F}_0$ has Hausdorff dimension 1, and $\mathcal{F}_\epsilon(L)$ is nowhere dense for all $L, \epsilon$.

The difficulty in counting geodesics on an arbitrary surface is that a closed geodesic on any surface follows simple closed curves, as will be explained shortly. There are exactly three simple closed geodesics on a pair of pants, and infinitely many simple closed geodesics on any other surface. Thus, closed geodesics on general surfaces are far more complex than closed geodesics on pairs of pants. The difficulty in extending the Birman Series type result for complete geodesics on an arbitrary surface is similar.

However, the study of geodesics on surfaces can be reduced to the study of curves on a pair of pants as follows (Figure 1). Let $\gamma \in \mathcal{G}$, and choose a starting point $\gamma(0)$ on $\gamma$. We can follow $\gamma$ around until we have traced out a subarc $\gamma_1$ with two self-intersection points. Then the graph of $\gamma_1$ looks like a figure 8. The two loops of this figure 8 are simple closed curves, and we can get a third simple closed curve by tracing the outside of these loops. It turns out that this figure 8 lies in a unique pair of pants $\mathcal{P}_1$. In this sense, $\gamma$ follows the three simple closed geodesics in $\mathcal{P}_1$.
We can keep going in this way to get a sequence $P_1, \ldots, P_n$ of pairs of pants containing subarcs $\gamma_1, \ldots, \gamma_n$ of $\gamma$. If we understand the sequence $P_1, \ldots, P_n$ of pairs of pants, then we have reduced the study of our closed geodesic to the study of geodesic arcs on pairs of pants. Because pairs of pants are simple multi-curves, there are already many tools available to deal with them. We hope to extend the above results to arbitrary surfaces using these techniques.

4. Future Questions

By studying geodesics on surfaces with bounds on both length and self-intersection number, I have been able to study geodesics with sparse self-intersections. These geodesics (with sparse self-intersections) are invisible from the point of view of ergodic theory on surfaces: Techniques from ergodic theory give us that almost all geodesics have maximally many self-intersections. In the future, I would like to study phenomenon that are invisible to ergodic theory in other contexts. For example,

- Let $S$ be a hyperbolic surface, and $\alpha$ a geodesic arc on $S$. By the ergodicity of the geodesic flow, the Birkhoff ergodic theorem implies that

$$i(\alpha, \gamma) \approx l(\alpha)l(\gamma)$$

for almost any closed geodesic $\gamma \in \mathcal{G}$. That is, the intersection number grows linearly in length of $\gamma$. What can be said about geodesics that intersect $\alpha$ sub-linearly with length?

References


