Example 1. Find the maximum and minimum values of the function \( f(x,y,z) = x^2 + y^2 + z^2 \) subject to the constraint \( x^4 + y^4 + z^4 = 1 \). (Exercise #11 in Stewart, §15.8)

Solution 1. Let \( g(x, y, z) = x^4 + y^4 + z^4 \). Then the constraint is \( g(x, y, z) = 1 \). Note that the level set \( g(x, y, z) = 1 \) is compact: a level set is compact if there is no way for a sequence of points in the level set to escape to infinity; if \( g(x, y, z) = 1 \) for some point \((x, y, z)\) then \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\) and \(-1 \leq z \leq 1\), so there is no way for a sequence of points in the level set to escape to infinity.

Since the domain is compact, we only need to find the constrained critical points of \( f \) and compare the critical values. We have

\[
\nabla f(x,y,z) = (2x, 2y, 2z) \\
\nabla g(x,y,z) = (4x^3, 4y^3, 4z^3).
\]

So the single-constraint Lagrange multiplier equation

\[
\nabla f(x,y,z) = \lambda \nabla g(x,y,z)
\]

becomes

\[
(2x, 2y, 2z) = \lambda (4x^3, 4y^3, 4z^3).
\]

We must find a simultaneous solution of the above vector equation and the constraint equation \( g(x, y, z) = 1 \). That is, we must solve the system of equations

\[
2x = 4\lambda x^3 \\
2y = 4\lambda y^3 \\
2z = 4\lambda z^3 \\
x^4 + y^4 + z^4 = 1
\]

in the four unknowns \( x, y, z, \lambda \). We can see immediately that \( \lambda \) can’t be zero, because then \( x, y, z \) would all be zero from the first three equations and
then $x^4 + y^4 + z^4$ would be 0, contradicting the last equation. We therefore know:

\[
\begin{align*}
x &= 0 \quad \text{or} \quad x^2 = \frac{1}{2\lambda} \\
y &= 0 \quad \text{or} \quad y^2 = \frac{1}{2\lambda} \\
z &= 0 \quad \text{or} \quad z^2 = \frac{1}{2\lambda} \\
x^4 + y^4 + z^4 &= 1.
\end{align*}
\]

Let’s enumerate the possibilities, based on how many of the coordinates are zero: it is impossible for $x = y = z = 0$ because $x^4 + y^4 + z^4 = 1$; if two of $x$, $y$, and $z$ are zero then the remaining one must be $\pm 1$ (e.g., if $x = y = 0$ then $z^4 = 1$ so $z = \pm 1$). Thus we have the following constrained critical points

\[
(0,0,\pm 1), (0,\pm 1,0), (\pm 1,0,0).
\]

If just one coordinate is zero, say $x = 0$ then we get

\[
y^2 = \frac{1}{2\lambda} = z^2
\]

so $y = \pm z$. Substituting this and $x = 0$ into $x^4 + y^4 + z^4 = 1$, we get $2y^4 = 1$ so $y = \pm \frac{1}{\sqrt{2}}$. This gives us the critical point $(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

Since the cases where $y$ (or $z$) is the coordinate that is zero are similar, we get the following constrained critical points:

\[
(0,\pm \frac{1}{\sqrt{2}},\pm \frac{1}{\sqrt{2}}), (\pm \frac{1}{\sqrt{2}},0,\pm \frac{1}{\sqrt{2}}), (\pm \frac{1}{\sqrt{2}},\pm \frac{1}{\sqrt{2}},0).
\]

Finally, we have to consider the possibility that none of the coordinates is zero. In that case, we get $x^2 = y^2 = z^2$, so $x^4 + y^4 + z^4 = 3x^4 = 1$. Therefore we have the constrained critical points

\[
(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}).
\]
Here is a complete list of the constrained critical points and critical values:

\[ f(0, 0, \pm 1) = 1 \]
\[ f(0, \pm 1, 0) = 1 \]
\[ f(\pm 1, 0, 0) = 1 \]
\[ f(0, \pm \frac{1}{\sqrt{4^2 + \sqrt{2}}} \cdot \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \]
\[ f(\pm \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = \sqrt{2} \]
\[ f(\pm \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \sqrt{2} \]
\[ f(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3} \]

Since the domain was compact, the maxima occur at the critical points where the value of \( f \) is largest, namely

\[ (\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}) \]

The minima occur at the critical points where the value of \( f \) is smallest,

\[ (0, 0, \pm 1), (0, \pm 1, 0), (\pm 1, 0, 0). \]

\[ \square \]

**Another solution.** Let \( f_1(a, b, c) = a + b + c \) and \( g_1(a, b, c) = a^2 + b^2 + c^2 \). Then \( f(x, y, z) = f_1(x^2, y^2, z^2) \) and \( g(x, y, z) = g_1(x^2, y^2, z^2) \). If we manage to find the extrema of \( f_1(a, b, c) \) subject to the constraints \( g_1(a, b, c) = 1, a \geq 0, b \geq 0, c \geq 0 \) then we will be able to recover the extrema of \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = 1 \) by \( x = \pm \sqrt{a}, y = \pm \sqrt{b}, \) and \( z = \pm \sqrt{c} \).

To find the extrema of \( f_1 \), solve the Lagrange multiplier problem:

\[ \nabla f_1(a, b, c) = \lambda \nabla g_1(a, b, c) \]
\[ g_1(a, b, c) = 1. \]

Since \( \nabla f_1(a, b, c) = (1, 1, 1) \) and \( \nabla g_1(a, b, c) = (2a, 2b, 2c) \), this becomes

\[ (1, 1, 1) = \lambda (2a, 2b, 2c) \]
\[ a^2 + b^2 + c^2 = 1 \]
\[ a \geq 0 \]
\[ b \geq 0 \]
\[ c \geq 0 \]

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Since $\lambda$ cannot be zero, we get

$$\frac{1}{2\lambda} = a = b = c$$

$$a^2 + b^2 + c^2 = 1$$

$$a \geq 0$$

$$b \geq 0$$

$$c \geq 0$$

Thus $3a^2 = 1$ so $a = b = c = \pm \frac{1}{\sqrt{3}}$. Only one of these solutions satisfies the additional constraint that none of the $a, b, c$ can be negative, so we get one constrained critical point $(a, b, c) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ with critical value $\sqrt{3}$.

We still have to consider the boundary cases, where $a = 0$, $b = 0$, or $c = 0$. If $a = 0$, the problem becomes to maximize $b + c$ subject to the constraints $b^2 + c^2 = 1$ and $b \geq 0$ and $c \geq 0$. Again doing Lagrange multipliers with one constraint, one gets

$$(1, 1) = \lambda(2b, 2c)$$

$$b^2 + c^2 = 1$$

$$b \geq 0$$

$$c \geq 0$$

Again, $\lambda$ can’t be zero, so we get $b = c = \pm \frac{1}{\sqrt{2}}$. Only one of these satisfies the inequalities $b \geq 0$, $c \geq 0$ so we get one more constrained critical point $(a, b, c) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ with critical value $\sqrt{2}$.

We must also include the boundary points $b = 0$ and $c = 0$ here. Therefore we get two additional candidates for extrema: $(a, b, c) = (0, 0, \pm 1)$ and $(a, b, c) = (0, \pm 1, 0)$. Since all of $a$, $b$, and $c$ must be non-negative, we can rule out two of these and we get $(a, b, c) = (0, 0, 1)$ and $(a, b, c) = (0, 1, 0)$.

We must repeat the above calculation for the boundary pieces $b = 0$ and $c = 0$ (instead of $a = 0$). The situation is symmetrical and we get the additional critical points

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$(1, 0, 0).$$
A complete list of candidates for extrema and the values of $f_1$ is therefore:

$$f_1\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \sqrt{3}$$

$$f_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \sqrt{2}$$

$$f_1\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

$$f_1\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

$$f_1(0, 0, 1) = 1$$

$$f_1(0, 1, 0) = 1$$

$$f_1(1, 0, 0) = 1$$

Therefore $f_1$ is maximized with $(a, b, c) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (which corresponds to $(x, y, z) = \left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$) and minimized with $(a, b, c) = (0, 0, 1)$ or $(a, b, c) = (0, 1, 0)$ or $(a, b, c) = (1, 0, 0)$ (which corresponds to $(x, y, z) = (0, 0, \pm 1)$ or $(x, y, z) = (0, \pm 1, 0)$ or $(x, y, z) = (\pm 1, 0, 0)$).

**Example 2.** Find the extrema of the function

$$f(x, y, z) = xyz^{1/3}$$

on the region

$$x + y + z = 1$$

$$x \geq -1$$

$$y \geq -2$$

$$z \geq -3.$$  

**Solution.** Here is a picture of the domain:
We will eliminate points from this region until we are down to a finite number of possible values for the extrema and a list of points where those values occur. The largest and smallest entries in that list will be the extrema.

Compute the gradient of \( f \):

\[
\nabla f(x, y, z) = (yz^{1/3}, xz^{1/3}, \frac{1}{3}xyz^{-2/3}).
\]

The first collection of points we can eliminate are the smooth interior points of the domain where \( \nabla f \) and which are not constrained critical points. From the above calculation, we see that \( \nabla f \) is not defined if \( z = 0 \) (because of the factor \( z^{-2/3} \) in \( \frac{\partial f}{\partial z} \)), so we will not be able to eliminate any point where \( z = 0 \). Since we will never be able to eliminate a point with \( z = 0 \) from consideration, we can always assume that \( z \neq 0 \) below.

**Step 1: Interior.** Let \( g(x, y, z) = x + y + z \). If \( z \neq 0 \), a constrained critical point will occur when

\[
\nabla f(x, y, z) = \lambda \nabla g(x, y, z)g(x, y, z) = 1.
\]

That is,

\[
(yz^{1/3}, xz^{1/3}, \frac{1}{3}xyz^{-2/3}) = \lambda(1, 1, 1)
\]

\[
x + y + z = 1
\]

or,

\[
yz^{1/3} = xz^{1/3} = \frac{1}{3}xyz^{-2/3} = \lambda
\]

\[
x + y + z = 1.
\]
The first equation tells us

\[ yz = xz = \frac{1}{3} xy. \]

Since \( z \neq 0 \) we can divide by \( z \) and we get \( y = x \). If \( x = y = 0 \) then \( z = 1 \) by the constraint, so we get one constrained critical point at \((0,0,1)\). Otherwise, we can divide by \( x \) or \( y \) in the equation displayed above and get \( z = \frac{1}{3} x = \frac{1}{3} y \) so \( x = y = 3z \). Therefore we get

\[ 3z + 3z + z = 1 \]

from the constraint, so \( z = \frac{1}{7} \) and \( x = y = \frac{3}{7} \). We have another constrained critical point at \((\frac{3}{7}, \frac{3}{7}, \frac{1}{7})\). Therefore, the candidates for extrema so far are

\[
\begin{align*}
(0,0,1) \\
(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}) \\
z = 0.
\end{align*}
\]

We can eliminate all other points that are not on the boundary of this region. Visually, our candidates for extrema are:

**Step 2: Boundary Piece I.** Now we have to consider the boundary. Let’s begin with the piece \( x = -1 \). We will do this piece with substitution. Let \( f_1(y,z) = f(-1,y,z) = -yz^{1/3} \). Let \( g_1(y,z) = g(-1,y,z) = -1 + y + z \).
Finding extrema of \( f \) subject to the constraints

\[
\begin{align*}
x + y + z &= 1 \\
x &= -1 \\
y &\geq -2 \\
z &\geq -3.
\end{align*}
\]

is the same as finding extrema of \( f_1 \) with the constraints

\[
\begin{align*}
g_1(y, z) &= 1 \\
y &\geq -2 \\
z &\geq -3.
\end{align*}
\]

We can eliminate any smooth points of this region where \( f_1 \) is defined and does not have a constrained critical point. That is, we solve

\[
\nabla f(y, z) = \lambda \nabla g(y, z)
\]

\[
i.e.,
\]

\[
(z^{1/3}, \frac{1}{3}yz^{-2/3}) = \lambda(1, 1)
\]

\[
y + z = 2.
\]

The first equation gives

\[
z^{1/3} = \frac{1}{3}yz^{-2/3}
\]

so \( y = 3z \). Substituting this into the other equation gives

\[
3z + z = 2
\]

hence \( z = \frac{1}{2} \) and \( y = \frac{3}{2} \). We obtain another constrained critical point

\[
(x, y, z) = (-1, \frac{3}{2}, \frac{1}{2}).
\]

We can now eliminate all points on the interior of this piece where \( \nabla f_1(y, z) \) is defined except this critical point. That is, we can eliminate all points on this line, except for \((-1, \frac{3}{2}, \frac{1}{2})\) and \((-1, 2, 0)\) since \( \nabla f_1 \) is not defined at \((2, 0)\).

Here is what the set of candidates for extrema looks like now: 
STEP 3: BOUNDARY PIECE II. Now consider the piece $y = -2$. This is very similar to the last piece. Making the substitution $y = -2$, we have $f_2(x, z) = f(x, -2, z) = -2xz^{1/3}$ and $g_2(x, z) = g(x, -2, z) = x - 2 + z$. The constraint is $g_2(x, z) = 1$.

The function $f_2(x, z)$ fails to be differentiable when $z = 0$, so we cannot eliminate the point $(3, -2, 0)$. To find the constrained critical points, we look at

$$\nabla f_2(x, z) = \lambda \nabla g_2(x, z)$$

$$g_2(x, z) = 1$$

i.e.,

$$\left( z^{1/3}, \frac{1}{3}xz^{-2/3} \right) = \lambda (1, 1)$$

$$x - 2 + z = 1.$$ 

From the first equation, $z^{1/3} = \frac{1}{3}xz^{-2/3}$ so $x = 3z$. From the second equation, $3z - 2 + z = 1$ so $z = \frac{3}{4}$ and $x = \frac{9}{4}$. We get the constrained critical point $\left( \frac{9}{4}, -2, \frac{3}{4} \right)$.

Now the candidates for extrema are the red regions below:
STEP 3: BOUNDARY PIECE III. We substitute \( z = -3 \) and get \( f_3(x, y, -3) = xy\sqrt{-3} \) and \( g_3(x, y) = g(x, y, -3) = x + y - 3 \). The Lagrange multiplier problem is

\[
\nabla f_3(x, y) = \lambda \nabla g_3(x, y)
\]

\[
g_3(x, y) = 1
\]

which translates to

\[
(y\sqrt{-3}, x\sqrt{-3}) = \lambda (1, 1)
\]

\[
x + y - 3 = 1.
\]

The first equation gives us \( y\sqrt{-3} = \lambda = x\sqrt{-3} \) so \( x = y \) and the second equation tells us \( x + y = 4 \) so we deduce \( x = y = 2 \). This gives us the constrained critical point \( (2, 2, -3) \). Since \( \nabla f \) is defined for all points on this piece of the domain and this piece is smooth, we can eliminate all points on the interior of this piece except for the constrained critical point. Now the collection of candidates for extrema looks like this:
Step 4: The piece $z = 0$. It is still possible that the extrema of $f$ could occur on the line $z = 0$. However, since $f(x, y, 0) = 0$ independent of $x$ and $y$, there is only one critical value to compare with the others. If it turns out to be the largest, the entire line will be a maximum and if it turns out to be the smallest, the entire line will be a minimum.

Step 5: Comparing the candidates. Here is our list of candidates and the values that $f$ takes at them:

$$
\begin{align*}
  &f(0, 0, 1) = 1 \quad \text{interior} \\
  &f\left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right) = \frac{9}{49} \frac{1}{\sqrt[7]{7}} = \frac{9}{7^{7/3}} \quad \text{interior} \\
  &f(x, y, 0) = 0 \quad \nabla f \text{ not defined} \\
  &f(-1, \frac{3}{2}, \frac{1}{2}) = -\frac{3}{2} \sqrt[2]{\frac{3}{2}} = -\frac{3}{2^{4/3}} \quad \text{boundary I} \\
  &f\left(\frac{9}{4}, -2, \frac{3}{4}\right) = -\frac{9}{2} \sqrt[4]{\frac{3}{4}} = -\frac{3^{7/3}}{2^{5/3}} = 6^{1/3} \frac{9}{4} \quad \text{boundary II} \\
  &f(2, 2, -3) = -4(3^{1/3}) \quad \text{boundary III} \\
  &f(-1, -2, 4) = (-1)(-2)(4^{1/3}) = 2^{5/3} \quad \text{intersection of boundaries I and II} \\
  &f(-1, 5, -3) = (-1)(5)(-3)^{1/3} = 5(3^{1/3}) \quad \text{intersection of boundaries I and III} \\
  &f(6, -2, -3) = 6(-2)(-3)^{1/3} = 12(3^{1/3}) \quad \text{intersection of boundaries II and III}.
\end{align*}
$$

Here are the values in order:

$$
-4(3^{1/3}) < 0 < \frac{9}{7^{7/3}} < 1 < \frac{3}{2^{4/3}} < 2^{5/3} < 6^{1/3} \frac{9}{4} < 5(3^{1/3}) < 12(3^{1/3}).
$$
Therefore the maximum occurs at \((6, -2, -3)\) where the value of \(f\) is \(12(3^{1/3})\) and the minimum occurs at \((2, 2, -3)\) where the value of \(f\) is \(-4(3^{1/3})\). \(\square\)