Continuity

Def: A function $f(x)$ is **continuous at** $x = a$ if the following three conditions all hold:

(1) $f(a)$ exists

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$.

So: A function $f(x)$ is **discontinuous at** $x = a$ if any one of (1)-(3) fails.

Types of Discontinuities: Removable, Jump, Essential.

Theorem 1: The following are continuous *at every point in their domains*:

- Polynomials
- Rational functions
- Exponentials & Logarithms
- Trig functions & Inverse trig functions

Note: Piecewise functions *may not be* continuous on their entire domains.

Example: The function $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ has a domain of $(-\infty, \infty)$, but it is not continuous at $x = 0$.

Theorem 2: Suppose both $f(x)$ and $g(x)$ are continuous at $x = a$. Then:

(a) $f + g$, $f - g$, and fg are continuous at $x = a$.

(b) If $g(a) \neq 0$, then f/g is continuous at $x = a$.

Theorem 3: If f is continuous at a, and if g is continuous at $f(a)$, then $f \circ g$ is continuous at a.

Intermediate Value Theorem (IVT): Suppose f is continuous on $[a, b]$.

If k is any number between $f(a)$ and $f(b)$, then there exists a number $c \in [a, b]$ such that $f(c) = k$.

This theorem is intuitive (easy to believe), but not obvious (it is hard to prove). The IVT is useful for proving that solutions to equations exist, but does not tell us what those solutions are!

Example: Using the IVT to Prove Existence of Solutions

Problem: Prove that $\cos x = x$ has a solution x between 0 and $\frac{\pi}{2}$.

Strategy: Let $f(x) = x - \cos x$. We *want to show* that there is some number $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$, because that will mean that $\cos(c) = c$.

Solution: Let $f(x) = x - \cos x$. Observe that $f(x)$ is continuous (because it is the difference of two continuous functions). Therefore, we can try to apply the IVT to $f(x)$ on the interval $[0, \frac{\pi}{2}]$. Let's do that.

Notice that

$$f(0) = 0 - \cos(0) = -1 < 0$$
$$f(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - \cos(\tfrac{\pi}{2}) = \tfrac{\pi}{2} > 0.$$

So:

$$f(0) < 0 < f(\tfrac{\pi}{2}).$$

Therefore, by the IVT (choosing $k = 0$), there exists $c \in (0, \frac{\pi}{2})$ with $f(c) = 0$, so $c - \cos(c) = 0$, so $\cos(c) = c$. This number c is our solution. $\Diamond$

Limits and Continuity

Intuition: The statement $\lim_{x\to a} f(x) = L$ means:

∘ Roughly: As x approaches a, the function values $f(x)$ approach L.

∘ More precisely: If x is *sufficiently close* to a, then the function values $f(x)$ can be made *arbitrarily close* to L.

Fact: The limit $\lim_{x\to a} f(x)$ exists $\iff$ $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$.

Def: A function $f(x)$ is **continuous at** $x = a$ if the following all hold:

(1) $f(a)$ exists
(2) $\lim_{x\to a} f(x)$ exists
(3) $\lim_{x\to a} f(x) = f(a)$.

So: A function $f(x)$ is **discontinuous at** $x = a$ if any one of (1)-(3) fails.

Limit Laws

Theorem L1: Suppose that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

(a) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
(b) $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
(c) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
(d) If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Theorem L2: If $f(x)$ is continuous on $\mathbb{R}$, and if $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a} f(g(x)) = f\Big(\lim_{x\to a} g(x)\Big).$$

Corollary: If $\lim_{x\to a} g(x)$ exists, then (for $n \in \mathbb{Z}^+$)

$$\lim_{x\to a}(g(x))^n = \Big(\lim_{x\to a} g(x)\Big)^n \qquad \text{and} \qquad \lim_{x\to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to a} g(x)}.$$

Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ for all x near a (except possibly for $x = a$), and if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Types of Discontinuities

Def: A function $f(x)$ is **continuous at** $x = a$ if the following all hold:

(1) $f(a)$ exists

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$.

So: A function $f(x)$ is **discontinuous at** $x = a$ if any one of (1)-(3) fails.

Def: Suppose $f(x)$ is *discontinuous* at $x = a$. The discontinuity is called:

Removable: If $\lim_{x \to a} f(x)$ exists. (That is: (2) holds, but (1) or (3) fails.)

Jump: If both $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ are finite, but not equal.

Essential: If one of $\lim_{x \to a-} f(x)$ or $\lim_{x \to a+} f(x)$ is infinite or does not exist.

Optional: The Definition of a Limit

Q: What is the precise meaning of $\lim_{x \to 2} f(x) = 5$?

The statement "As x approaches 2, the function values $f(x)$ approach 5" is vague and imprecise. Let's clarify it.

Idea: "If x is *sufficiently close* to 2, then the function values $f(x)$ can be made *arbitrarily close* to 5." Let's clarify this even further:

Precisely: For any (arbitrarily small) open interval $(5 - \epsilon, 5 + \epsilon)$ around $y = 5$, there is a (sufficiently small) open interval $(2 - \delta, 2 + \delta)$ around $x = 2$ such that: If $x \in (2 - \delta, 2 + \delta)$, then $f(x) \in (5 - \epsilon, 5 + \epsilon)$.

In general, the mathematical definition of "limit" is as follows:

Def: The statement $\lim_{x \to a} f(x) = L$ means:

"For every open interval $(L - \epsilon, L + \epsilon)$ around $y = L$, there exists an open interval $(a - \delta, a + \delta)$ around $x = a$ such that:

If $x \in (a - \delta, a + \delta)$, then $f(x) \in (L - \epsilon, L + \epsilon)$."

This is the technical definition that one uses to establish the limit laws, the theorems about continuity, the Intermediate Value Theorem, etc.

Vertical Asymptotes

Def: A line $x = a$ is a **vertical asymptote** of $f(x)$ if *any* of the following holds:

$$\lim_{x\to a-} f(x) = -\infty \qquad \text{or} \qquad \lim_{x\to a+} f(x) = -\infty \qquad \text{or}$$
$$\lim_{x\to a-} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to a+} f(x) = \infty.$$

Again: If any one of these holds, then $x = a$ is a vertical asymptote.

Horizontal Asymptotes

Def: A line $y = b$ is a **horizontal asymptote** of $f(x)$ if any of the following holds:

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

So: A function can have 0, 1, or 2 horizontal asymptotes.

Examples:

- $f(x) = \frac{1}{x^n}$, for $n > 0$, has a horizontal asymptote $y = 0$:
$$\lim_{x\to-\infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x\to+\infty} \frac{1}{x^n} = 0.$$
- $g(x) = a^x$, for $a > 1$, has a horizontal asymptote $y = 0$:
$$\lim_{x\to-\infty} a^x = 0 \qquad \text{whereas} \qquad \lim_{x\to\infty} a^x = \infty.$$

Careful: If instead $0 < a < 1$, then $\lim_{x\to-\infty} a^x = \infty$, whereas $\lim_{x\to\infty} a^x = 0$.

Indeterminate Forms

Indeterminate Forms: The following expressions are "indeterminate":

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad 0\cdot\infty, \qquad \infty - \infty.$$

N.B.: The following are *not* indeterminate. Here, $a, b \in \mathbb{R}$ with $b \neq 0$.

$$\frac{b}{0}, \qquad \frac{a}{\infty}, \qquad b\cdot\infty, \qquad a+\infty,\ a-\infty.$$

Derivatives: Introduction

Def: The **average rate of change** of $f(x)$ **on the interval** $[a, a+h]$ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

Geometrically: This is the slope of the secant line to $y = f(x)$ on $[a, a+h]$.

Def: The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** of $f(x)$ at $x = a$:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Geometrically: This is the slope of the tangent line to $y = f(x)$ at $x = a$.

The equation of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is (from Point-Slope Formula):

$$y - f(a) = m(x - a).$$

We now know that $m = f'(a)$.

Derivatives as Functions

We can talk about the derivative at any point x:

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

That is: The derivative $f'(x)$ is a function giving the slope of the tangent line to $y = f(x)$ at $(x, f(x))$.

Linearity Property: For differentiable functions $f(x)$, $g(x)$ and constants $c \in \mathbb{R}$:

$$\frac{d}{dx}[cf(x)] = c\,\frac{d}{dx}f(x)$$
$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Derivatives: An Alternate Formula

Recall: The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** of $f(x)$ at $x = a$:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad (\star)$$

Interpretations: The derivative of $f(x)$ at $x = a$ describes the:
- ◦ Instantaneous rate of change of $f(x)$ at $x = a$.
- ◦ Slope of tangent line to $f(x)$ at $x = a$.
- ◦ Best linear approximation to $f(x)$ at $x = a$.

In the formula $(\star)$, set $x = a + h$. Then $h \to 0$ means $x \to a$. This gives an alternate definition of the derivative at $x = a$:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

This formula for the derivative is sometimes useful.

Differentiability (Introduction)

The derivative is defined in terms of the limit $(\star)$. This limit *may or may not exist*, meaning that derivatives may or may not exist. This leads to:

Def: A function $f(x)$ is **differentiable at** $x = a$ if its derivative $f'(a)$ exists. That is:

$f(x)$ is differentiable at $x = a$ $\iff$ The limit $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

N.B.: The converse is false! Continuous does not imply differentiable.
Again: Differentiable functions are continuous. Not the other way around.

Failure of Differentiability:
- ◦ Discontinuities (Removable, Jump, Essential)
- ◦ Cusps
- ◦ Vertical tangent lines

Derivatives of Common Functions

Derivative of Power Functions:

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Derivative of Exponential and Logarithmic Functions:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}\ln|x| = \frac{1}{x}$$
$$\frac{d}{dx}(a^x) = a^x \ln a \qquad \frac{d}{dx}\log_a(x) = \frac{1}{x \ln a}.$$

Derivatives of Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Derivatives of Inverse Trig Functions:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

Differentiation Laws

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))\, g'(x).$$

Differentiability

The derivative of $f(x)$ at $x = a$ is defined in terms of a limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit *may or may not exist*, meaning that the derivative at $x = a$ may or may not exist. This leads to:

Def: A function $f(x)$ is **differentiable at** $x = a$ if its derivative $f'(a)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

Q: Which functions are differentiable? Here are two important theorems.

Theorem: Let $f(x)$, $g(x)$ be differentiable at $x = a$. Then:
(a) $f + g$ and $f - g$ and fg are differentiable at $x = a$.
(b) If $g(a) \neq 0$, then f/g is differentiable at $x = a$.
(c) $f \circ g$ and $g \circ f$ are differentiable at $x = a$.

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

The converse is false! Differentiable functions are continuous, not the other way around. Again: A continuous function may or may not be differentiable.

Failure of Differentiability:
- Discontinuities (Removable, Jump, Essential)
- Cusps
- Vertical tangent lines

Examples:
(a) $f(x) = |x|$ is continuous on $\mathbb{R}$, but has a cusp at $x = 0$.
(b) $g(x) = \sqrt[3]{x^2}$ is continuous on $\mathbb{R}$, but has a cusp at $x = 0$.
(c) $h(x) = \sqrt[3]{x}$ is continuous on $\mathbb{R}$, but has a vertical tangent line at $x = 0$.

Linear Approximation

Suppose $f(x)$ is differentiable at $x = a$. Its derivative at $x = a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

So, when $h \approx 0$ is small, we have

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

and therefore:

$$f(a+h) \approx f(a) + f'(a)h.$$

Moral: Given $f(a)$ and $f'(a)$, we can approximate $f(a+h)$ for small h.

Linear Approximation: Another View

Here is another point of view. The derivative of $f(x)$ at $x = a$ is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

So, when $x \approx a$, we have

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

and therefore:

$$f(x) \approx f(a) + f'(a)(x - a).$$

Moral: Near $x = a$, the function f is approximately its tangent line at $x = a$.

Def: Suppose $f(x)$ is differentiable at $x = a$.

The **linear approximation** (or **linearization**) to $f(x)$ **at** $x = a$ is the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

The above says: For $x \approx a$, we have

$$f(x) \approx L(x).$$

Moral: Near $x = a$, the function f is approximately the linear function L. The graph $y = L(x)$ is the tangent line to $f(x)$ at $x = a$.

Optional: Quadratic Approximation

Approximating a complicated function $f(x)$ by a *linear* function $L(x)$ is rather crude. Better idea: Approximate $f(x)$ by a *quadratic* function $Q(x)$.

Def: Suppose $f(x)$ is twice-differentiable at $x = a$.

The **quadratic approximation** to $f(x)$ **at** $x = a$ is the quadratic function

$$Q(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

Although it is not obvious to us at the moment, this function Q turns out to be the quadratic that best approximates f. So, for $x \approx a$, we have:

$$f(x) \approx Q(x).$$

Moral: Near $x = a$, the function f is approximately the quadratic function Q. The graph $y = Q(x)$ is the "tangent parabola" to $f(x)$ at $x = a$.

Remark: Higher-Order Approximation

As one might guess, there is also a notion of cubic approximation (involving 3rd derivatives), quartic approximation (involving 4th derivatives), etc.

Geometrically: There are corresponding notions of "tangent cubic" and "tangent quartic," etc.

These ideas lead to the notion of a *Taylor series*, which is typically the last topic in a year-long calculus course. We won't pursue this today.

Example: Using Linear Approximation

Problem: Approximate $(2.01)^5$ using linear approximation. Is your approximation larger or smaller than the actual value?

Idea: The number $(2.01)^5$ is close to 2^5, which we know how to calculate. This suggests taking $f(x) = x^5$ and $a = 2$ and $h = 0.01$.

Solution: Let $f(x) = x^5$, so $f'(x) = 5x^4$. Let $a = 2$ and $h = 0.01$. The formula for linear approximation is

$$\begin{aligned} f(a+h) &\approx f(a) + f'(a)h \\ f(2+0.01) &\approx f(2) + f'(2)(0.01). \end{aligned}$$

Since $f(2) = 2^5 = 32$ and $f'(2) = 5(2)^4 = 80$, we have

$$f(2.01) \approx 32 + (80)(0.01) = 32 + 0.8 = \boxed{32.800}$$

Since the graph of $f(x) = x^5$ is concave up, it lies above its tangent lines. Therefore, our approximation of 32.800 is $\boxed{\text{smaller than the actual value.}}$

Fun fact: The actual value is roughly 32.808. Our approximation is pretty close!

Mean-Value Theorem

Mean-Value Theorem (MVT): If $f(x)$ is a differentiable function on $[a, b]$, then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is: There is a point $(c, f(c))$ on the graph $y = f(x)$ such that: The tangent line at $(c, f(c))$ has slope equal to that of the secant line on $[a, b]$.

Why is this important? Answer: The MVT is the main tool used in proving several major theorems in calculus. Two of these are listed below.

Consequences

Def: Let $f(x)$ be a function on an interval I.

- $f(x)$ **increasing on** I: If $x_1, x_2 \in I$ with $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- $f(x)$ **decreasing on** I: If $x_1, x_2 \in I$ with $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Def: We say $x = c$ is a **turning point** of $f(x)$ if: At $x = c$, the function $f(x)$ changes from increasing to decreasing, or from decreasing to increasing.

Theorem: Suppose that $f(x)$ is differentiable on an interval I.
(a) If $f'(x) > 0$ on I, then f is increasing on I.
(b) If $f'(x) < 0$ on I, then f is decreasing on I.

Theorem: Suppose that $f(x)$ is differentiable on an interval I. If $f'(x) = 0$ on I, then $f(x)$ is constant on I.

Curve Sketching: Overview

To sketch the graph of $y = f(x)$, we need to identify:

(a) Vertical and horizontal asymptotes of f, if any.
(b) Intervals on which f is increasing/decreasing, and any turning points.
(c) Local maxima and local minima of f.
(d) Intervals on which f is concave up/down, and any inflection points.

The First Derivative

Def: Let $f(x)$ be a function on an interval I.

- $f(x)$ **increasing on** I: If $x_1, x_2 \in I$ with $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- $f(x)$ **decreasing on** I: If $x_1, x_2 \in I$ with $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Def: We say $x = c$ is a **turning point** of $f(x)$ if: At $x = c$, the function $f(x)$ changes from increasing to decreasing, or from decreasing to increasing.

Theorem: Suppose that $f(x)$ is differentiable on an interval I.

(a) If $f'(x) > 0$ on I, then f is increasing on I.
(b) If $f'(x) < 0$ on I, then f is decreasing on I.

The First Derivative Test: Suppose that $f(x)$ is differentiable near $x = c$. If $f'(c) = 0$ or $f'(c)$ d.n.e., then:

(a) If f' changes from $+$ to $-$ at $x = c$, then $x = c$ is a local max.
(b) If f' changes from $-$ to $+$ at $x = c$, then $x = c$ is a local min.
(c) If f' does not change sign at $x = c$, then $x = c$ is not a local max or min.

The Second Derivative

Def: Let $f(x)$ be a (differentiable) function on an interval I.

- $f(x)$ **concave up on** I: The graph of $f(x)$ lies above its tangent lines.
- $f(x)$ **concave down on** I: The graph of $f(x)$ lies below its tangent lines.

Def: We say $x = c$ is an **inflection point** of $f(x)$ if: At $x = c$, the function $f(x)$ changes from concave up to down, or from concave down to up.

Theorem: Suppose that $f(x)$ is twice-differentiable on an interval I.

(a) If $f''(x) > 0$ on I, then f is concave up on I.
(b) If $f''(x) < 0$ on I, then f is concave down on I.

The Second Derivative Test: Suppose that $f''(x)$ is continuous near $x = c$.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a local min.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a local max.

Optimization: Theory

Def: Let $f(x)$ be a function.

- We say $x = c$ is a **local max of** f if: $f(c) \geq f(x)$ for all x near c.
- We say $x = c$ is a **local min of** f if: $f(c) \leq f(x)$ for all x near c.

Def: Let $f(x)$ be a function on an interval I.

- Say $x = c$ is an **absolute max of** f **on** I if: $f(c) \geq f(x)$ for all $x \in I$.
- Say $x = c$ is an **absolute min of** f **on** I if: $f(c) \leq f(x)$ for all $x \in I$.

Example: The function $f(x) = x^3$ has no absolute max/min on $(-1, 1)$. However, $f(x) = x^3$ has both an absolute max and an absolute min on $[-1, 1]$.

Example: The function $f(x) = \tan(\frac{\pi}{2}x)$ has no absolute max/min on $[-1, 1]$.

Extreme Value Theorem (EVT): If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ has an absolute max and an absolute min on $[a, b]$.

Optimization: Method

Def: A **critical point** of $f(x)$ is a number $x = c$ in the domain of $f(x)$ such that: $f'(c) = 0$ or $f'(c)$ d.n.e.

Theorem: If $x = c$ is a local max/min of f, then $x = c$ is a critical point of f.

Warning: The converse is false! Not every critical point is a local max or min. (For example, it could be an inflection point.)

Method: Given a continuous function $f(x)$ on a closed interval $[a, b]$. To find the absolute max/min values of $f(x)$ on $[a, b]$:

(1) Calculate the values $f(c)$ for all critical points $c \in (a, b)$.
(2) Calculate $f(a)$ and $f(b)$.

Conclusion: The largest of the values in Steps 1-2 is the absolute max of $f(x)$ on $[a, b]$. The smallest is the absolute min of $f(x)$ on $[a, b]$.

L'Hopital's Rule

L'Hopital's Rule applies to limits of the form $0/0$ and ∞/∞.

L'Hopital's Rule 1: Suppose f and g are differentiable, and that $g'(x) \neq 0$ near $x = a$ (except possibly at $x = a$). If

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

L'Hopital's Rule 2: Suppose f and g are differentiable, and that $g'(x) \neq 0$ near $x = a$ (except possibly at $x = a$). If

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Indeterminate Forms

Indeterminate Forms: The following symbols are "indeterminate":

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^\infty \qquad 0^0 \qquad \infty^0$$

A limit taking any of these indeterminate forms could potentially be any real number, or possibly $+\infty$ or $-\infty$.

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \qquad \frac{\infty}{0} \qquad \frac{1}{\infty} \qquad 1\cdot\infty \qquad \infty+\infty \qquad 1+\infty \qquad 0^\infty$$