Continuity

Def: A function $f(x)$ is continuous at $x = a$ if the following three conditions all hold:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$.

So: A function $f(x)$ is discontinuous at $x = a$ if any one of (1)-(3) fails.

Types of Discontinuities: Removable, Jump, Essential.

Theorem 1: The following are continuous at every point in their domains:

- Polynomials
- Rational functions
- Exponentials & Logarithms
- Trig functions & Inverse trig functions

Note: Piecewise functions may not be continuous on their entire domains.

Example: The function $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ has a domain of $(-\infty, \infty)$, but it is not continuous at $x = 0$.

Theorem 2: Suppose both $f(x)$ and $g(x)$ are continuous at $x = a$. Then:

a) $f + g$, $f - g$, and $fg$ are continuous at $x = a$.

b) If $g(a) \neq 0$, then $f/g$ is continuous at $x = a$.

Theorem 3: If $f$ is continuous at $a$, and if $g$ is continuous at $f(a)$, then $f \circ g$ is continuous at $a$.

Intermediate Value Theorem (IVT): Suppose $f$ is continuous on $[a, b]$.
If $k$ is any number between $f(a)$ and $f(b)$, then there exists a number $c \in [a, b]$ such that $f(c) = k$.

This theorem is intuitive (easy to believe), but not obvious (it is hard to prove). The IVT is useful for proving that solutions to equations exist, but does not tell us what those solutions are!
Example: Using the IVT to Prove Existence of Solutions

Problem: Prove that \( \cos x = x \) has a solution \( x \) between 0 and \( \frac{\pi}{2} \).

Strategy: Let \( f(x) = x - \cos x \). We want to show that there is some number \( c \in (0, \frac{\pi}{2}) \) such that \( f(c) = 0 \), because that will mean that \( \cos(c) = c \).

Solution: Let \( f(x) = x - \cos x \). Observe that \( f(x) \) is continuous (because it is the difference of two continuous functions). Therefore, we can try to apply the IVT to \( f(x) \) on the interval \( [0, \frac{\pi}{2}] \). Let’s do that.

Notice that
\[
\begin{align*}
f(0) &= 0 - \cos(0) = -1 < 0 \\
f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0.
\end{align*}
\]

So:
\[
f(0) < 0 < f\left(\frac{\pi}{2}\right).
\]

Therefore, by the IVT (choosing \( k = 0 \)), there exists \( c \in (0, \frac{\pi}{2}) \) with \( f(c) = 0 \), so \( c - \cos(c) = 0 \), so \( \cos(c) = c \). This number \( c \) is our solution. \( \diamond \)
Limits and Continuity

Intuition: The statement \( \lim_{x \to a} f(x) = L \) means:
- Roughly: As \( x \) approaches \( a \), the function values \( f(x) \) approach \( L \).
- More precisely: If \( x \) is sufficiently close to \( a \), then the function values \( f(x) \) can be made arbitrarily close to \( L \).

Fact: The limit \( \lim_{x \to a} f(x) \) exists \iff \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \).

Def: A function \( f(x) \) is continuous at \( x = a \) if the following all hold:
1. \( f(a) \) exists
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \).

So: A function \( f(x) \) is discontinuous at \( x = a \) if any one of (1)-(3) fails.

Limit Laws

Theorem L1: Suppose that both \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Then:
(a) \( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \)
(b) \( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \).
(c) \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \).
(d) If \( \lim_{x \to a} g(x) \neq 0 \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \).

Theorem L2: If \( f(x) \) is continuous on \( \mathbb{R} \), and if \( \lim_{x \to a} g(x) \) exists, then
\[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right). \]

Corollary: If \( \lim_{x \to a} g(x) \) exists, then (for \( n \in \mathbb{Z}^+ \))
\[ \lim_{x \to a} (g(x))^n = \left( \lim_{x \to a} g(x) \right)^n \quad \text{and} \quad \lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}. \]

Squeeze Theorem: If \( f(x) \leq g(x) \leq h(x) \) for all \( x \) near \( a \) (except possibly for \( x = a \)), and if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} h(x) = L \), then \( \lim_{x \to a} g(x) = L \).
Types of Discontinuities

Def: A function \( f(x) \) is **continuous at** \( x = a \) if the following all hold:

1. \( f(a) \) exists
2. \( \lim_{x \to a} f(x) \) exists
3. \( \lim_{x \to a} f(x) = f(a) \).

So: A function \( f(x) \) is **discontinuous at** \( x = a \) if any one of (1)-(3) fails.

Def: Suppose \( f(x) \) is **discontinuous** at \( x = a \). The discontinuity is called:

Removable: If \( \lim_{x \to a} f(x) \) exists. (That is: (2) holds, but (1) or (3) fails.)

Jump: If both \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) are finite, but not equal.

Essential: If one of \( \lim_{x \to a^-} f(x) \) or \( \lim_{x \to a^+} f(x) \) is infinite or does not exist.

Optional: The Definition of a Limit

Q: What is the precise meaning of \( \lim_{x \to 2} f(x) = 5 \)?

The statement “As \( x \) approaches 2, the function values \( f(x) \) approach 5” is vague and imprecise. Let’s clarify it.

Idea: “If \( x \) is **sufficiently close** to 2, then the function values \( f(x) \) can be made **arbitrarily close** to 5.” Let’s clarify this even further:

Precisely: For any (arbitrarily small) open interval \((5 - \epsilon, 5 + \epsilon)\) around \( y = 5 \), there is a (sufficiently small) open interval \((2 - \delta, 2 + \delta)\) around \( x = 2 \) such that: If \( x \in (2 - \delta, 2 + \delta) \), then \( f(x) \in (5 - \epsilon, 5 + \epsilon) \).

In general, the mathematical definition of “limit” is as follows:

Def: The statement \( \lim_{x \to a} f(x) = L \) means:

“For every open interval \((L - \epsilon, L + \epsilon)\) around \( y = L \), there exists an open interval \((a - \delta, a + \delta)\) around \( x = a \) such that:

If \( x \in (a - \delta, a + \delta) \), then \( f(x) \in (L - \epsilon, L + \epsilon) \).”

This is the technical definition that one uses to establish the limit laws, the theorems about continuity, the Intermediate Value Theorem, etc.
Vertical Asymptotes

Def: A line $x = a$ is a **vertical asymptote** of $f(x)$ if any of the following holds:

\[
\lim_{{x \to a^-}} f(x) = -\infty \quad \text{or} \quad \lim_{{x \to a^+}} f(x) = -\infty \quad \text{or} \quad \lim_{{x \to a^-}} f(x) = \infty \quad \text{or} \quad \lim_{{x \to a^+}} f(x) = \infty.
\]

Again: If any one of these holds, then $x = a$ is a vertical asymptote.

Horizontal Asymptotes

Def: A line $y = b$ is a **horizontal asymptote** of $f(x)$ if any of the following holds:

\[
\lim_{{x \to \infty}} f(x) = b \quad \text{or} \quad \lim_{{x \to -\infty}} f(x) = b.
\]

So: A function can have 0, 1, or 2 horizontal asymptotes.

Examples:

- $f(x) = \frac{1}{x^n}$, for $n > 0$, has a horizontal asymptote $y = 0$:

  \[
  \lim_{{x \to -\infty}} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{{x \to \infty}} \frac{1}{x^n} = 0.
  \]

- $g(x) = a^x$, for $a > 1$, has a horizontal asymptote $y = 0$:

  \[
  \lim_{{x \to -\infty}} a^x = 0 \quad \text{whereas} \quad \lim_{{x \to \infty}} a^x = \infty.
  \]

  Careful: If instead $0 < a < 1$, then $\lim_{{x \to -\infty}} a^x = \infty$, whereas $\lim_{{x \to \infty}} a^x = 0$.

Indeterminate Forms

Indeterminate Forms: The following expressions are “indeterminate”:

\[
0, \quad \infty, \quad 0 \cdot \infty, \quad \infty - \infty.
\]

N.B.: The following are *not* indeterminate. Here, $a, b \in \mathbb{R}$ with $b \neq 0$.

\[
\frac{b}{0}, \quad \frac{a}{\infty}, \quad b \cdot \infty, \quad a + \infty, \quad a - \infty.
\]