Laplace Transform: Examples

Def: Given a function $f(t)$ defined for $t > 0$. Its **Laplace transform** is the function, denoted $F(s) = \mathcal{L}\{f\}(s)$, defined by:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

(*Issue:* The Laplace transform is an improper integral. So, does it always exist? i.e.: Is the function $F(s)$ always finite? *Answer:* This is a little subtle. We'll discuss this next time.)

Fact (Linearity): The Laplace transform is **linear**:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\, \mathcal{L}\{f_1(t)\} + c_2\, \mathcal{L}\{f_2(t)\}.$$

Example 1: $\mathcal{L}\{1\} = \dfrac{1}{s}$

Example 2: $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

Example 3: $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2 + a^2}$

Example 4: $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2 + a^2}$

Example 5: $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$

Useful Fact: Euler's Formula says that

$$\begin{aligned} e^{it} &= \cos t + i \sin t \\ e^{-it} &= \cos t - i \sin t \end{aligned}$$

Therefore,

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

Laplace Transform: Key Properties

Recall: Given a function $f(t)$ defined for $t > 0$. Its **Laplace transform** is the function, denoted $F(s) = \mathcal{L}\{f\}(s)$, defined by:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Notation: In the following, let $F(s) = \mathcal{L}\{f(t)\}$.

Fact A: We have

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Fact B (Magic): Derivatives in $t \to$ Multiplication by s (well, almost).

$$\mathcal{L}\{f'(t)\} = \begin{pmatrix} s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(s) \\ -f(0) \end{pmatrix} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(s) \\ -f(0) \\ -f'(0) \end{pmatrix} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = \begin{pmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \\ 1 \end{pmatrix} \cdot \begin{pmatrix} F(s) \\ -f(0) \\ \cdots \\ -f^{(n-2)}(0) \\ -f^{(n-1)}(0) \end{pmatrix}$$

$$= s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Fact C (Magic): Multiplication by $t \to$ Derivatives in s (almost).

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

Laplace Transform: Existence

Recall: Given a function $f(t)$ defined for $t > 0$. Its **Laplace transform** is the function defined by:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Issue: The Laplace transform is an improper integral. So, does it always exist? i.e.: Is the function $F(s)$ always finite?

Def: A function $f(t)$ is of **exponential order** if there is a threshold $M \geq 0$ and constants $K > 0$, $a \in \mathbb{R}$ such that

$$|f(t)| \leq Ke^{at}, \qquad \text{when } t \geq M.$$

Equivalently: There is a threshold $M \geq 0$ and a constant $a \in \mathbb{R}$ such that the function $\frac{f(t)}{e^{at}}$ is **bounded** when $t \geq M$ (meaning that $\left|\frac{f(t)}{e^{at}}\right| \leq K$).

Theorem: Let $f(t)$ be a function that is:
(1) continuous;
(2) of exponential order (with exponent a).
Then:
(a) $F(s) = \mathcal{L}\{f(t)\}(s)$ exists for all $s > a$; and
(b) $\lim_{s\to\infty} F(s) = 0$.

Example: The function $f(t) = \exp(t^2)$ is not of exponential order.

Remark: If $f(t)$ is not continuous, or not of exponential order, then the Laplace transform may or may not exist.

Inverse Laplace Transform: Existence

Want: A notion of "inverse Laplace transform." That is, we would like to say that if $F(s) = \mathcal{L}\{f(t)\}$, then $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Issue: How do we know that $\mathcal{L}$ even has an inverse $\mathcal{L}^{-1}$? Remember, not all operations have inverses.

To see the problem: imagine that there are different functions $f(t)$ and $g(t)$ which have the same Laplace transform $H(s) = \mathcal{L}\{f\} = \mathcal{L}\{g\}$. Then $\mathcal{L}^{-1}\{H(s)\}$ would make no sense: after all, should $\mathcal{L}^{-1}\{H\}$ be $f(t)$ or $g(t)$?

Fortunately, this bad scenario can never happen:

Theorem: Let $f(t), g(t)$ be continuous functions on $[0, \infty)$ of exponential order. If $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f(t) = g(t)$ for all $t \in [0, \infty)$.

Def: Let $f(t)$ be continuous on $[0, \infty)$ and of exponential order.

We call $f(t)$ the **inverse Laplace transform** of $F(s) = \mathcal{L}\{f(t)\}$. We write $f = \mathcal{L}^{-1}\{F\}$.

Fact (Linearity): The inverse Laplace transform is **linear:**

$$\mathcal{L}^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1\, \mathcal{L}^{-1}\{F_1(s)\} + c_2\, \mathcal{L}^{-1}\{F_2(s)\}.$$

Inverse Laplace Transform: Examples

Example 1: $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$

Example 2: $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^n}\right\} = e^{at}\dfrac{t^{n-1}}{(n-1)!}$

Example 3: $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+b^2}\right\} = \cos bt$

Example 4: $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+b^2}\right\} = \dfrac{1}{b}\sin bt$

Fact A: We have $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$.

Therefore:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\, \mathcal{L}^{-1}\{F(s)\}.$$

Partial Fractions

Setup: Given a rational function

$$R(x) = \frac{p(x)}{q(x)}.$$

Saying that R is rational means that both p and q are polynomials.

Begin by factoring the denominator $q(x)$ over $\mathbb{R}$. (The phrase "over $\mathbb{R}$" means, e.g., that $x^3 + 4x$ factors as $x(x^2 + 4)$. That is, we do not allow complex numbers. Factoring into $x(x + 2i)(x - 2i)$ would be factoring "over $\mathbb{C}$.")

Case 1: $q(x)$ has linear distinct factors, meaning that we can express $q(x) = (x - a_1) \cdots (x - a_n)$. In this case, we write

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - a_1} + \cdots + \frac{A_n}{x - a_n}.$$

Case 2: $q(x)$ has linear factors, where some are repeated. Corresponding to this factor like $(x - a)^p$, we write

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_p}{(x - a)^p}.$$

Case 3: $q(x)$ has a quadratic factor, not repeated. Corresponding to a factor like $(x - (\mu + i\nu))(x - (\mu - i\nu)) = (x - \mu)^2 + \nu^2$, we write

$$\frac{A(x - \mu) + B\nu}{(x - \mu)^2 + \nu^2}.$$

Case 4: $q(x)$ has repeated quadratic factors. Corresponding to a factor like $((x - \mu)^2 + \nu^2)^n$, we write

$$\frac{A_1(x - \mu) + B_1\nu}{(x - \mu)^2 + \nu^2} + \frac{A_2(x - \mu) + B_2\nu}{((x - \mu)^2 + \nu^2)^2} + \cdots + \frac{A_n(x - \mu) + B_n\nu}{((x - \mu)^2 + \nu^2)^n}.$$

Example: Here is a partial fraction decomposition:

$$\frac{7x^3 + 2}{(x - 3)^2(x^2 + 25)^2} = \frac{A}{x - 3} + \frac{B}{(x - 3)^2} + \frac{Cx + 5D}{x^2 + 25} + \frac{Ex + 5F}{(x^2 + 25)^2}.$$

Review: Intro to Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

It can be thought of as an "infinite polynomial."

The number x_0 is called the **center**.

A power series may converge for some values of x, but diverge for other values of x. A power series will *always* converge at its center $x = x_0$ (do you see why?).

Question: Given a power series, for what values of x does it converge, and for what values of x does it diverge?

Theorem: Given a power series $\sum a_n(x-x_0)^n$. Then either:

(i) The power series converges only at $x = x_0$. (Case $R = 0$)

(ii) The power series converges for all $x \in \mathbb{R}$. (Case $R = +\infty$)

(iii) The power series converges on an interval $|x - x_0| < R$, and diverges if $|x - x_0| > R$.

The number R is called the **radius of convergence**.

Note: This theorem says nothing about the convergence/divergence at the endpoints of the interval. Those have to be checked separately.

Finding the Interval of Convergence:

(1) Determine the center x_0.

(2) Determine the radius of convergence R. Use the Ratio Test to do this.

(3) Check convergence/divergence at the endpoints.

Review: Power Series are Functions

Given a power series $\sum a_n(x-x_0)^n$, we can think of it as a function of x:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Domain of $f(x)$: The set of all x-values for which the series converges. That is, the domain is exactly the interval of convergence.

Although every power series (with $R > 0$) is a function, not all functions arise in this way. i.e.: Not all functions are equal to a convergent power series! Those functions which are have a special name:

Def: A function $f(x)$ is **analytic** at $x = x_0$ if it is equal to a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

that has *positive* radius of convergence $R > 0$.

Analytic functions are the best-behaved functions in all of calculus. For example, every analytic function is infinitely-differentiable:

Theorem: Let $f(x)$ be analytic at x_0, say $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ with radius of convergence $R > 0$. Then:

(a) f is infinitely-differentiable on the interval $(x_0 - R, x_0 + R)$.

(b) The derivative of f is:

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} \qquad (*)$$

(c) The indefinite integral of f is:

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1} \qquad (\dagger)$$

(d) The radii of convergence of $(*)$ and $(\dagger)$ are both R.

Note: Again, this theorem says nothing about convergence/divergence at the endpoints. Those have to be checked separately.

Review: Taylor Series

Recall that a **power series** is any series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

Def: Let $f(x)$ be infinitely-differentiable on an interval $|x-x_0| < R$.

The **Taylor series of f at $x = x_0$** is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

So, by definition: every Taylor series is a power series.

Conversely, every power series with $R > 0$ is a Taylor series:

Theorem: If $f(x) = \sum a_n(x-x_0)^n$ is a power series with radius of convergence $R > 0$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

(So: The given power series $\sum a_n(x-x_0)^n$ is exactly the Taylor series of $f(x)$.)

Corollary: If $f(x)$ is analytic at x_0, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

So: If f is analytic at $x = x_0$, then the Taylor series of f does converge to f.

There **are** functions infinitely-differentiable at x_0 but **not** analytic at x_0. For those functions, the Taylor series at x_0 will only equal $f(x)$ at $x = x_0$ – even if the Taylor series converges on an interval $(x_0 - R, x_0 + R)$!

Classic Scary Example: The function $f(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ is infinitely-differentiable on all of $\mathbb{R}$. Its derivatives at $x = 0$ are $f^{(n)}(0) = 0$ for each $n = 0, 1, 2, \ldots$. Therefore, its Taylor series at $x = 0$ is

$$0 + \tfrac{0}{1!}x + \tfrac{0}{2!}x^2 + \cdots$$

which converges on all of $\mathbb{R}$ to the function $g(x) = 0$.

Point: Our function $f(x)$ (which is defined on all of $\mathbb{R}$) is only equal to its Taylor series (which is also defined on all of $\mathbb{R}$) at $x = 0$. Weird!

Review: Examples of Taylor Series

Many of the functions we care about are analytic, meaning that they are equal to a power series. For example:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

It's also good to know about

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Again: If $f(x)$ is analytic, meaning that it is equal to a power series, then that power series is the Taylor series of $f(x)$.

Review: Taylor Polynomials

Def: The **nth-degree Taylor polynomial of** $f(x)$ is the polynomial

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Main Point of Taylor Series: The Taylor polynomials $T_n(x)$ are the best polynomial approximations to $f(x)$ near the center $x = x_0$.

Example: The first Taylor polynomial

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

is the best linear approximation near $x = x_0$. After all, $T_1(x)$ is exactly the tangent line to $f(x)$ at $x = x_0$.

Similarly: The second Taylor polynomial

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

is the "tangent parabola" at $x = x_0$. Et cetera.

Series Solutions to ODEs: Ordinary Points

Goal: Given a 2nd-order linear ODE (with non-constant coefficients)

$$y'' + p(x)y' + q(x)y = 0.$$

We usually cannot solve for $y(x)$ explicitly.

Hope: Maybe we can express $y(x)$ as a power series: $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$.

If we can do this, then the partial sums (i.e.: the Taylor polynomials) are **polynomial approximations** to $y(x)$.

Def: Given a 2nd-order linear ODE

$$y'' + p(x)y' + q(x)y = 0.$$

A point x_0 is a **ordinary point** if both $p(x), q(x)$ are analytic at x_0.
A point x_0 is a **singular point** otherwise.

Theorem: Given a 2nd-order linear ODE

$$y'' + p(x)y' + q(x)y = 0.$$

Suppose that x_0 is an ordinary point.

Then the general solution may be written as a power series

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x),$$

where a_0, a_1 are arbitrary constants, and $y_1(x), y_2(x)$ are power-series solutions (and hence analytic at x_0). Moreover, $\{y_1(x), y_2(x)\}$ is a fundamental set of solutions.

Also: the radii of convergence for the series solutions of $y_1(x), y_2(x)$ are at least the minimum of the radii of convergence of the series for $p(x)$ and $q(x)$.

Fourier Series: Intro

Recall: A **power series** is an "infinite polynomial."

Given a function $f(x)$, the **Taylor series of** $f(x)$ is a power series where the coefficients are determined by some formula ($a_n = \frac{f^{(n)}(x_0)}{n!}$).

Often, the Taylor series of $f(x)$ does converge to $f(x)$. So, Taylor series let us approximate $f(x)$ by a sequence of polynomials.

Today: A **trigonometric series** is an "infinite trig polynomial."

Given a function $f(x)$, the **Fourier series of** $f(x)$ is a trigonometric series where the coefficients are determined by some formula (below).

Often, the Fourier series of $f(x)$ does converge to $f(x)$. So, Fourier series let us approximate $f(x)$ by a sequence of "waves."

Def: A **trigonometric series** is an infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right).$$

Given a function $f(x)$, its **Fourier series** is the trigonometric series whose coefficients are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad \text{(A)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx. \qquad \text{(B)}$$

So, every Fourier series is a trigonometric series. Conversely, every convergent trigonometric series is a Fourier series:

Fact: If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$ is a trigonometric series which converges, then its coefficients a_n, b_n are given by (A)-(B) above.

Fourier Convergence Theorem: Suppose $f(x)$ is periodic of period $2L$. Suppose also that $f(x)$ and $f'(x)$ are piecewise-continuous on $[-L, L)$.

Then the Fourier series of f converges to $f(x)$ at all points where f is continuous. It converges to $\frac{1}{2}[f(x+) + f(x-)]$ at all points where f is discontinuous.

Linear Algebra Review: Orthogonal Bases

Recall: A basis $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ for a subspace $V \subset \mathbb{R}^n$ is **orthogonal** if:

- The basis vectors are mutually orthogonal: $\mathbf{w}_i \cdot \mathbf{w}_j = 0$ (for $i \neq j$).

Given a basis $\mathcal{B}$, it is generally a pain to find the $\mathcal{B}$-coordinates of a given vector. But when $\mathcal{B}$ is an *orthogonal* basis, there is a very simple formula:

Fact: Let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be an orthogonal basis for a subspace $V \subset \mathbb{R}^n$.

Then every vector $\mathbf{y} \in V$ can be written:

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{w}_k}{\mathbf{w}_k \cdot \mathbf{w}_k}\mathbf{w}_k$$

This is sometimes called the **Fourier expansion** of $\mathbf{y} \in V$. Hmm.....

Fourier Series: A Powerful Geometric Perspective

Def: The **inner product** of two functions $f(x), g(x)$ on an interval $[a, b]$ is:

$$(f, g) = \int_a^b f(x)g(x)\,dx.$$

We say that $f(x)$ and $g(x)$ are **orthogonal functions** if $(f, g) = 0$.

A set of functions $f_1(x), f_2(x), \dots$ is an **orthogonal set** of functions if every pair in the set is orthogonal: $(f_i, f_j) = 0$ for $i \neq j$.

Fact: Consider the functions

$$v_n(x) = \cos\Big(\frac{n\pi}{L}x\Big), \qquad w_n(x) = \sin\Big(\frac{n\pi}{L}x\Big).$$

Then the set of functions $\{v_1, v_2, \dots, w_1, w_2, \dots\}$ is an orthogonal set.

Observation: Let $f(x)$ be a function. Its Fourier coefficients are exactly

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi}{L}x\Big)\,dx = \frac{(f, v_n)}{(v_n, v_n)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,dx = \frac{(f, w_n)}{(w_n, w_n)}.$$

Therefore, the Fourier series of $f(x)$ is just

$$f(x) = \frac{(f, v_0)}{(v_0, v_0)}v_0 + \sum_{n=1}^{\infty}\frac{(f, v_n)}{(v_n, v_n)}v_n + \sum_{n=1}^{\infty}\frac{(f, w_n)}{(w_n, w_n)}w_n$$

This is an (infinite-dimensional) analogue of the linear algebra above!

Useful for Fourier Series: Even and Odd Functions

Facts:

- ∘ Even function $\times$ Even function $=$ Even function
- ∘ Odd function $\times$ Odd function $=$ Even function
- ∘ Odd function $\times$ Even function $=$ Odd function.
- ∘ If $f(x)$ is odd, then $\displaystyle\int_{-L}^{L} f(x) = 0$.
- ∘ If $f(x)$ is even, then $\displaystyle\int_{-L}^{L} f(x) = 2\int_{0}^{L} f(x)$.

N.B.: A function $f(x)$ can be even, odd, neither, or both. Most functions are neither. The only function that is both even and odd is $f(x) \equiv 0$.

Intro to PDEs

ODE: Differential equation whose solutions $u = u(t)$ are functions of one variable. Derivatives involved are ordinary derivatives u' or u'' or u''', etc.

Initial-value Problem: Diff eqn + Initial values specified

PDE: Differential equation whose solutions $u = u(x, y)$ are functions of two variables (or more). Derivatives involved are partial derivatives u_x, u_y or u_{xx}, u_{xy}, u_{yy}, etc.

Dirichlet Problem: Diff eqn + Boundary values specified

Neumann Problem: Diff eqn + "Normal directional derivatives" specified.

Recall: Differential equations (both ODEs and PDEs) are classified by their **order**: i.e., the highest-order derivative appearing in the equation.

1st-Order PDEs: Most 1st-order PDEs can be converted into a 1st-order (nonlinear) ODE system.

This is called the "method of characteristics." We won't study it. The point is that 1st-order PDEs reduce to the study of 1st-order ODE systems.

2nd-Order PDEs: Our understanding of 2nd-order PDEs is largely based around understanding three foundational examples:

- Laplace Equation: $u_{xx} + u_{yy} = 0$
- Heat Equation: $u_{xx} = u_y$
- Wave Equation: $u_{xx} - u_{yy} = 0$.

These three equations are very different from each other. We'll only talk about the Laplace equation.

Example Problems: Solve the Laplace equation $u_{xx} + u_{yy} = 0$ subject to:

(D1) Dirichlet conditions on a Rectangle
(D2) Dirichlet conditions on the Interior of a Disk
(D3) Dirichlet conditions on the Exterior of a Disk
(D4) Dirichlet conditions on a Circular Sector
(D5) Dirichlet conditions on a Semi-infinite Strip
(N1) Neumann conditions on a Rectangle
(N2) Neumann conditions on the Interior of a Disk

We'll discuss problems (D1) and (D2). Problems (D3) and (D4) are HW.

Review: Circular Trigonometric Functions

Recall: The functions cos and sin are defined by

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Notice that **cos is even**, while **sin is odd**.

Fact 1: The Taylor series of cos and sin centered at $x = 0$ are

$$\cos x = \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad \sin x = \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Fact 2: The functions cos and sin satisfy the following **pythagorean identity**:

$$\cos^2(x) + \sin^2(x) = 1.$$

Corollary: The parametric curve given by $x = \cos(t)$, $y = \sin(t)$ is a **unit circle**.

Fact 3: The derivatives of cos and sin are:

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

Thus, both cos and sin solve the 2nd-order ODE given by $\boxed{y'' = -y}$

New: Hyperbolic Trigonometric Functions

Recall: The functions cosh and sinh are defined by

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x}), \quad \sinh x = \frac{1}{2}(e^{x} - e^{-x}).$$

Notice that **cosh is even**, while **sinh is odd**.

Fact 1: The Taylor series of cosh and sinh centered at $x = 0$ are

$$\cosh x = \sum_{0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad \sinh x = \sum_{0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

Fact 2: The functions cosh and sinh satisfy the following **pythagorean identity**

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Corollary: The parametric curve given by $x = \cosh(t), y = \sinh(t)$ is a **hyperbola**.

Fact 3: The derivatives of cosh and sinh are:

$$\frac{d}{dx}(\sinh x) = \cosh x, \qquad \frac{d}{dx}(\cosh x) = \sinh x.$$

Notice how there are no minus signs! That is, both sinh and cosh solve $\boxed{y'' = y}$

Laplace Equation: Dirichlet Problem for Rectangles: I

Goal: Solve the Laplace equation $u_{xx} + u_{yy} = 0$ on the rectangle $(0,a) \times (0,b)$ subject to the Dirichlet boundary conditions:

$$u(x,0) = 0 \qquad\qquad u(0,y) = 0$$
$$u(x,b) = 0 \qquad\qquad u(a,y) = f(y),$$

where $f(y)$ is a given function on $0 \leq y \leq b$.

Step 1: Assume that there is a non-trivial solution of the form $u(x,y) = X(x)Y(y)$. (Here, "non-trivial" means $u(x,y) \not\equiv 0$.) We have to find $X(x)$ and $Y(y)$.

If $u(x,y) = X(x)Y(y)$ solves $u_{xx} + u_{yy} = 0$, then

$$0 = u_{xx} + u_{yy} = X''(x)Y(y) + X(x)Y''(y) = XY\left(\frac{X''}{X} + \frac{Y''}{Y}\right).$$

Therefore:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

This last equation has a function of x equal to a function of y. Therefore, both sides must equal some constant. To obtain solutions which are *non-trivial*, this constant must be positive, so call it λ^2:

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2.$$

Therefore, we have initial-value problems:

$$X'' - \lambda^2 X = 0 \qquad\qquad Y'' + \lambda^2 Y = 0$$
$$X(0) = 0 \qquad\qquad Y(0) = 0$$
$$Y(b) = 0.$$

Step 2: Solve the IVPs.

First, the general solution of the ODE for $X(x)$ is

$$X(x) = c_1\cosh(\lambda x) + c_2\sinh(\lambda x).$$

The initial condition $X(0) = 0$ gives $c_1 = 0$, and hence

$$X(x) = c_2\sinh(\lambda x).$$

Second, the general solution to the ODE for $Y(y)$ is

$$Y(y) = c_3\sin(\lambda y) + c_4\sin(\lambda y).$$

The initial condition $Y(0) = 0$ gives $c_3 = 0$, and hence

$$Y(y) = c_4\sin(\lambda y).$$

The initial condition $Y(b) = 0$ gives $c_4\sin(\lambda b) = 0$. We assume $c_4 \neq 0$, so that

$$\sin(\lambda b) = 0 \implies \lambda b = n\pi \text{ for any } n \in \mathbb{Z}$$
$$\implies \lambda = \frac{n\pi}{b} \text{ for any } n \in \mathbb{Z}.$$

We conclude that

$$\boxed{X(x) = c_2\sinh\left(\frac{n\pi}{b}x\right)} \qquad \boxed{Y(y) = c_4\sin\left(\frac{n\pi}{b}y\right)}.$$

Laplace Equation: Dirichlet Problem for Rectangles: II

Goal: Solve the Laplace equation $u_{xx} + u_{yy} = 0$ on the rectangle $(0, a) \times (0, b)$ subject to the Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= 0 \qquad\qquad & u(0,y) &= 0 \\ u(x,b) &= 0 & u(a,y) &= f(y), \end{aligned}$$

where $f(y)$ is a given function on $0 \leq y \leq b$.

Summary: We assumed that there is a solution of the form $u(x, y) = X(x)Y(y)$.

By requiring $u_{xx} + u_{yy} = 0$, together with *three of the four* boundary conditions, we were led to the functions

$$\begin{aligned} X_n(x) &= \sinh\left(\frac{n\pi}{b}x\right) \\ Y_n(y) &= \sin\left(\frac{n\pi}{b}y\right), \qquad n \in \mathbb{N}. \end{aligned}$$

So, our first conclusion is that the functions

$$u_n(x,y) = X_n(x)Y_n(y) = \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

solve both the PDE and *three of the four* boundary conditions, for any positive integer $n \in \mathbb{N}$.

Step 3: Finally, we need to impose the fourth boundary condition $u(a, y) = f(y)$. For this, we need a little trick.

Since the functions $u_1, u_2, u_3, \ldots$ all satisfy the PDE and the homogeneous boundary conditions, it follows that any finite linear combination $c_1u_1 + c_2u_2 + \cdots + c_Nu_N$ does, too. It turns out that an "infinite linear combination" also does, as well. That is, we consider

$$\boxed{u(x,y) = \sum_1^\infty c_n u_n(x,y) = \sum_1^\infty c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right).}$$

Step 4: We need to determine the coefficients c_n that make $u(a, y) = f(y)$ true. So, we require that the condition $u(a, y) = f(y)$ hold:

$$f(y) = u(a,y) = \sum_1^\infty \underbrace{c_n \sinh\left(\frac{n\pi}{b}a\right)}_{\text{call this } B_n} \sin\left(\frac{n\pi}{b}y\right) = \sum_1^\infty B_n \sin\left(\frac{n\pi}{b}y\right).$$

This is a Fourier series for $f(y)$! Therefore, the coefficients c_n are determined by the formula

$$\begin{aligned} c_n \sinh\left(\frac{n\pi}{b}a\right) = B_n &= \frac{1}{b}\int_{-b}^{b} f(y)\sin\left(\frac{n\pi}{b}y\right)dy \\ &= \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy. \qquad \text{(integrand is even)} \end{aligned}$$

Therefore,

$$\boxed{c_n = \frac{1}{\sinh\left(\frac{na\pi}{b}\right)} \cdot \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy.}$$

Laplace Equation: Dirichlet Problem for Disk Interior: I

Goal: Solve the Laplace equation $u_{xx} + u_{yy} = 0$ on the disk $\{x^2 + y^2 < a^2\}$ subject to Dirichlet boundary conditions.

Preliminaries: Polar Coordinates.

The Laplace equation in polar coordinates (r, θ) is:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

In polar coordinates, the interior of the disk is the region $0 \leq r < a$.

A Dirichlet boundary condition means specifying $u(r, \theta)$ on the boundary circle $r = a$:

$$u(a, \theta) = f(\theta),$$

where $f(\theta)$ is a *periodic function* of period 2π.

We can now try to mimic the steps in the case of a rectangle.

Note: We will require that $u(r, \theta)$ be a *bounded function* (this will be important later).

Step 1: Assume there is a non-trivial solution of the form $u(r, \theta) = R(r)\Theta(\theta)$, where $\Theta(\theta)$ is periodic of period 2π. We have to find $R(r)$ and $\Theta(\theta)$.

If $u(r, \theta) = R(r)\Theta(\theta)$ solves $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, then

$$0 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Therefore:

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

This last equation has a function of r equal to a function of θ. Therefore, both sides must equal some constant, say λ:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Therefore, we have ODEs:

$$r^2R'' + rR' - \lambda R = 0 \qquad\qquad \Theta'' + \lambda\Theta = 0.$$

The separation constant λ must be non-negative (see textbook), so we write $\lambda = \mu^2$. Thus:

$$r^2R'' + rR' - \mu^2R = 0 \qquad\qquad \Theta'' + \mu^2\Theta = 0.$$

Step 2: Solve the ODEs.

The 2nd-order linear ODE for $\Theta(\theta)$ is constant-coefficient, so

$$\Theta(\theta) = c_1\cos(\mu\theta) + c_2\sin(\mu\theta).$$

For $\Theta(\theta)$ to have period 2π, we need μ to be an integer – i.e.: $\mu = n \in \mathbb{Z}^+$.

The 2nd-order linear ODE for $R(r)$ is a Cauchy-Euler equation (appeared in HW 8). It has the general solution

$$R(r) = c_3r^n + c_4r^{-n}.$$

For $u(r, \theta)$ to be bounded, we need $R(r)$ bounded on $[0, a]$. So, for $n \geq 0$, we need $c_4 = 0$:

$$R(r) = c_3r^n.$$

Laplace Equation: Dirichlet Problem for Disk Interior: II

Goal: Solve the Laplace equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ on the disk $\{r < a\}$ subject to the Dirichlet boundary condition $u(a, \theta) = f(\theta)$.

Summary: We assumed that there is a (bounded) solution of the form $u(r, \theta) = R(r)\Theta(\theta)$, where Θ has period 2π.

By requiring $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, together with the requirements that $u(r, \theta)$ be bounded and $\Theta(\theta)$ be 2π-periodic, we were led to the functions

$$\begin{aligned} R_n(r) &= r^n \\ \Theta_n(\theta) &= c_3 \cos(n\theta) + c_4 \sin(n\theta). \end{aligned}$$

So, our first conclusion is that the functions

$$\begin{aligned} u_n(r, \theta) &= r^n \cos(n\theta) \\ v_n(r, \theta) &= r^n \sin(n\theta), \qquad n \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Step 3: Finally, we need to impose the boundary condition $u(a, \theta) = f(\theta)$. For this, we need the same superposition trick as for the rectangle.

That is: We consider the "infinite linear combination"

$$u(r, \theta) = \frac{c_0}{2}u_0(r, \theta) + \frac{k_0}{2}v_0(r, \theta) + \sum_1^\infty [c_n u_n(r, \theta) + k_n v_n(r, \theta)].$$

Since $u_0(r, \theta) \equiv 1$ and $v_0(r, \theta) \equiv 0$, we have:

$$\boxed{u(r, \theta) = \frac{c_0}{2} + \sum_1^\infty [c_n r^n \cos(n\theta) + k_n r^n \sin(n\theta)].}$$

Step 4: We need to determine the coefficients c_n, k_n that make $u(a, \theta) = f(\theta)$ true. So, we require that $u(a, \theta) = f(\theta)$ hold:

$$f(\theta) = u(a, \theta) = \frac{c_0}{2} + \sum_1^\infty [c_n a^n \cos n\theta + k_n a^n \sin n\theta].$$

This is a Fourier series for $f(\theta)$! Therefore, the coefficients c_n, k_n are determined by the formulas

$$\boxed{a^n c_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta}$$

$$\boxed{a^n k_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta.}$$

For Clarification: Cauchy-Euler Equations

Def: A **Cauchy-Euler equation** is a 2nd-order linear ODE of the form

$$x^2y'' + pxy' + qy = 0, \tag{CE}$$

where $p, q \in \mathbb{R}$ are constants.

Notice that $x = 0$ is a regular singular point of this ODE.

We learned how to solve Cauchy-Euler equations (HW #8). The trick was to make the substitutions

$$\begin{aligned} t &= \ln x \\ u(t) &= y(e^t). \end{aligned}$$

You showed that these substitutions transform (CE) into

$$u'' + (p-1)u' + qu = 0. \tag{$\star$}$$

This is a 2nd-order linear ODE with *constant coefficients*! Yay!

We solve ($\star$) by writing its characteristic equation

$$\lambda^2 + (p-1)\lambda + q\lambda = 0.$$

There are three possibilities:

(1) Real, distinct roots (λ_1, λ_2):

$$\begin{aligned} y(e^t) &= u(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} \\ \implies & \boxed{y(x) = c_1x^{\lambda_1} + c_2x^{\lambda_2}.} \qquad (x > 0) \end{aligned}$$

(2) Real, repeated roots (λ):

$$\begin{aligned} y(e^t) &= u(t) = c_1e^{\lambda t} + c_2\, te^{\lambda t} \\ \implies & \boxed{y(x) = c_1x^{\lambda} + c_2\, x^{\lambda}\ln(x).} \qquad (x > 0) \end{aligned}$$

(3) Complex conjugate roots ($\mu \pm i\nu$):

$$\begin{aligned} y(e^t) &= u(t) = e^{\mu t}\left[c_1\cos(\nu t) + c_2\sin(\nu t)\right] \\ \implies & \boxed{y(x) = x^{\mu}[c_1\cos(\nu\ln(x)) + c_2\sin(\nu\ln(x))]\,.} \qquad (x > 0) \end{aligned}$$