Variation of Parameters (for $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$)

Goal: Solve 1st-order linear systems $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.

Fact: Let $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ be a fundamental set of the ODE system $\mathbf{x}' = A\mathbf{x}$. Let $\mathbf{x}_p(t)$ be a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.

Then the general solution to $\mathbf{x}' = A\mathbf{x}' + \mathbf{g}(t)$ is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t).$$

Example: If A is invertible and $\mathbf{g}(t) \equiv \mathbf{b}$ is constant, then one particular solution is the equilibrium solution: i.e., we can take $\mathbf{x}_p(t) = -A^{-1}\mathbf{b}$.

Q: What if A is not invertible, or if $\mathbf{g}(t)$ is not constant? How to find $\mathbf{x}_p(t)$?

Theorem (Variation of Parameters): Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a fundamental set for $\mathbf{x}' = A\mathbf{x}$, with fundamental matrix

$$X(t) = \begin{pmatrix} | & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) \\ | & | \end{pmatrix}.$$

Then a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ is

$$\boxed{\mathbf{x}_p(t) = X(t) \int X^{-1}(t)\, \mathbf{g}(t)\, dt.}$$

Remark: Both the Fact above and the Theorem here are actually true for **all** 1st-order linear systems $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$. Yay! But in that level of generality, a fundamental set of solutions to $\mathbf{x}' = P(t)\mathbf{x}$ may be hard to find.

We will stick to the case of $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.

For Curiosity: Where did this formula come from?

Idea: Look for particular solutions $\mathbf{x}_p(t)$ of the form

$$\begin{aligned}\mathbf{x}_p(t) &= u_1(t)\mathbf{x}_1(t) + u_2(t)\mathbf{x}_2(t) \\ &= \begin{pmatrix} | & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) \\ | & | \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ &= X(t)\mathbf{u}(t)\end{aligned}$$

for some functions $\mathbf{u}(t) = (u_1(t), u_2(t))$ to be determined.

If $\mathbf{x}_p(t) = X(t)\mathbf{u}(t)$ really is a solution, then it makes $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ true, meaning

$$(X(t)\mathbf{u}(t))' = A(X(t)\mathbf{u}(t)) + \mathbf{g}(t),$$

so that (product rule)

$$X'\mathbf{u} + X\mathbf{u}' = AX\mathbf{u} + \mathbf{g},$$

so that (since $X' = AX$)

$$AX\mathbf{u} + X\mathbf{u}' = AX\mathbf{u} + \mathbf{g},$$

and hence (after canceling):

$$X\mathbf{u}' = \mathbf{g}.$$

Therefore, $\mathbf{u}'(t) = X^{-1}(t)\,\mathbf{g}(t)$, so that

$$\mathbf{u}(t) = \int X^{-1}(t)\,\mathbf{g}(t)\,dt.$$

We conclude that

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = X(t)\int X^{-1}(t)\,\mathbf{g}(t)\,dt.$$

1st-Order ODEs: Abstract Existence-Uniqueness

Q: Do all 1st-order **linear** ODEs have solutions? Almost:

Existence-Uniqueness (1st-Order: Linear): Let $p(t)$ and $g(t)$ be continuous functions on an open interval I. Let $t_0 \in I$ and $y_0 \in \mathbb{R}$.

Then there exists a unique function $y = \phi(t)$ that solves the initial-value problem

$$\begin{cases} y' + p(t)y = g(t), \\ y(t_0) = y_0 \end{cases}$$

on the interval I.

Point: Solutions *exist* (meaning there is at least one solution) and are *unique* (meaning there is exactly one).

Q: What about 1st-order ODEs that are non-linear? Is there an existence-uniqueness theorem that covers **all** 1st-order ODEs? Not quite, but almost:

Existence-Uniqueness (1st-Order: General): Let $f(t, y)$ be a function. Suppose that both f and $\partial f/\partial y$ are continuous on some rectangle $(a, b)\times(c, d)$ in the ty-plane. Let (t_0, y_0) be a point in the rectangle $(a, b) \times (c, d)$.

Then there exists a unique solution $y = \phi(t)$ to the initial-value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

on some interval $(t_0 - h, t_0 + h)$ inside of (a, b).

Note: In this general theorem, we only get existence and uniqueness on *some* interval inside of (a, b).

Note: Uniqueness $\Longrightarrow$ Graphs of solutions cannot intersect.

Classic Example: Consider the initial-value problem

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0. \end{cases}$$

The general theorem **does not apply** to this problem: $f(t, y) = y^{1/3}$ has $\partial f/\partial y = \frac{1}{3}y^{-2/3}$, which is **not** continuous on any rectangle containing $(0, 0)$.

1st-Order ODEs: Approximation Methods

Setup: Given a 1st-order IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

Suppose $y = \phi(t)$ is the solution (which you don't have a formula for).

Goal: Approximate $\phi(t)$ for values of t close to t_0.

Euler's Method

Choose a step size Δt. Define points $t_1 = t_0 + \Delta t$ and $t_2 = t_1 + \Delta t$, etc. The approximation to $\phi(t_{n+1})$ is

$$\boxed{y_{n+1} = y_n + \Delta t \, f(t_n, y_n).}$$

Geometry: There are two different ways to think about Euler's method:

(1) Riding along the direction field lines.

(2) Approximating the area under the curve $\phi'(t) = f(t, \phi(t))$ using a left rectangle.

Remark: The approximate solution that Euler's Method constructs is a *piecewise-linear* curve.

Improved Euler's Method

Choose a step size Δt. Define points $t_1 = t_0 + \Delta t$ and $t_2 = t_1 + \Delta t$, etc. The approximation to $\phi(t_{n+1})$ is

$$\begin{aligned} y_{n+1} &= y_n + \frac{\Delta t}{2}\,[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})], \quad \text{where} \\ y^*_{n+1} &= y_n + \Delta t \, f(t_n, y_n). \end{aligned}$$

Geometry: Again, there are two ways to think about this method:

(1) Riding along "*averaged*" direction field lines.

(2) Approximating the area under the curve $\phi'(t) = f(t, \phi(t))$ using a trapezoid.

For Clarification: Bernoulli Equations

Def: A **Bernoulli equation** is a 1st-order ODE of the form

$$y' + p(t)y = q(t)y^n. \tag{B}$$

If $n \geq 2$, then the Bernoulli equation (B) is nonlinear.

Fact: If $y(t)$ is a solution of the Bernoulli ODE (B), then $\nu = y^{1-n}$ is a solution of the linear ODE

$$\frac{1}{1-n}\nu' + p(t)\nu = q(t). \tag{L}$$

Point: The substitution $\nu = y^{1-n}$ converts your Bernoulli ODE (B) into a linear ODE (L), which can then solve (using integrating factors) for $\nu(t)$.

So: Bernoulli equations are just linear ODE in disguise!

For Clarification: Riccati Equations

Def: A **Riccati equation** is a 1st-order ODE of the form

$$y' = A(t)y^2 + B(t)y + C(t). \tag{R}$$

If $A(t) \not\equiv 0$, then the Riccati equation (R) is nonlinear.

Fact: If $y(t)$ and $y_0(t)$ are two solutions of the Riccati equation (R), then $\nu = y - y_0$ is a solution of the Bernoulli equation

$$\nu' + \left(-2A(t)y_0(t) - B(t)\right)\nu = A(t)\nu^2. \tag{$\star$}$$

Point: The substitution $\nu = y - y_0$ converts your Ricatti ODE (R) into a Bernoulli ODE ($\star$).

So: If you happen to know one solution $y_0(t)$ to the Riccati ODE, then you can setup the Bernoulli equation ($\star$) and solve it for $\nu(t)$ using the method above. You now have the general solution $y(t) = \nu(t) + y_0(t)$ to (R).

1st-Order Linear Systems: Existence-Uniqueness

We have discussed existence-uniqueness theorems for 1st-order single equations. What about for 1st-order linear ODE systems?

Existence-Uniqueness (1st-Order Linear Systems): Let $g_1(t), g_2(t)$ and $p_{11}(t), p_{12}(t), p_{21}(t), p_{22}(t)$ be continuous on an open interval I. Let $t_0 \in I$, and let $x_0, y_0 \in \mathbb{R}$.

Then there exists a unique solution $(x(t), y(t))$ to the initial-value problem

$$\begin{cases} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

on the interval I.

1st-Order ODE: Examples of Bifurcations

HW #5: Examines 1st-order autonomous ODE $y' = f(a, y)$ that involve a parameter a. As the parameter a changes, the number of equilibrium solutions may change, and their stability may change, too.

Values of a at which the equilibrium solutions change in quantity or in quality are called **bifurcation points**.

Bifurcations can be understood through **bifurcation diagrams**, which are sketches in the ay-plane.

There are many, many kinds of bifurcations. In this class, we will only look at a few simple ones:

1. Saddle bifurcations.
Model Examples: $y' = a + y^2$ and $y' = a - y^2$.

2. Transcritical bifurcations.
Model Example: $y' = y(a - y)$.

3. Pitchfork bifurcations.
Model Examples: $y' = y(a + y^2)$ and $y' = y(a - y^2)$.

2nd-Order Linear ODE: General Observations

Def: A **second-order ODE** is a differential equation

$$y'' = F(t, y, y').$$

A second-order ODE is **linear** if it is of the form

$$y'' + p(t)y' + q(t)y = g(t).$$

It is **homogeneous** if $g(t) \equiv 0$. Otherwise, it is **non-homogeneous**.

Key Fact: 2nd-Order Linear ODE → 1st-Order Linear ODE Systems

Here's how: Given a 2nd-order linear ODE $y'' + p(t)y' + q(t)y = g(t)$. Let us define $x_1 = y$ and $x_2 = y'$. Then:

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -p(t)y' - q(t)y + g(t) \\ &= -p(t)x_2 - q(t)x_1 + g(t) \end{aligned}$$

That is:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

This is a 1st-order linear ODE system.

Notice that $x_1(t) = y(t)$ is the solution to our original 2nd-order ODE.

Existence-Uniqueness (2nd-Order Linear): Let $p(t), q(t), g(t)$ be continuous functions on an open interval I. Let $t_0 \in I$, and let $y_0, y_1 \in \mathbb{R}$.

Then there exists a unique function $y = \phi(t)$ that solves the initial-value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

on the interval I.

2nd-Order Linear ODE: Homogeneous, Const-Coeff.

Recall: A second-order linear ODE has the form

$$y'' + p(t)y' + q(t)y = g(t).$$

For now, this is too difficult. So, we simplify things with two assumptions:

(1) Homogeneous: This is the condition that $g(t) \equiv 0$.

$$y'' + p(t)y' + q(t)y = 0.$$

(2) Constant Coefficient: This is the special case

$$y'' + by' + cy = 0.$$

That is, $p(t) \equiv b$ and $q(t) \equiv c$ are both constants.

Observation: Start with the 2nd-order ODE $y'' + by' + cy = 0$.

Transform it into a 1st-order linear ODE system by setting $x_1 = y$ and $x_2 = y'$. This gives:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$ is exactly

$$\boxed{p(\lambda) = \lambda^2 + b\lambda + c}$$

So: Eigenvalues of $\begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$ are the roots of $p(\lambda) = \lambda^2 + b\lambda + c$.

Corollary: Let λ_1, λ_2 be roots of the **characteristic equation**

$$\lambda^2 + b\lambda + c = 0.$$

Then the general solution to $y'' + by + c = 0$ is:

(a) If $\lambda_1 \neq \lambda_2$ are real, distinct:

$$\boxed{y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

(b) If $\lambda_1 = \lambda_2 = \lambda$ are real, repeated:

$$\boxed{y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}$$

(c) If $\lambda_1, \lambda_2 = \mu \pm i\nu$ are complex conjugates:

$$\boxed{y = c_1 e^{\mu t}\cos(\nu t) + c_2 e^{\mu t}\sin(\nu t)}$$

Application: Spring-Mass Systems

Physics: The equation of motion for a spring-mass system, possibly with damping, possibly with external forcing, is:

$$my'' + \gamma y' + ky = F(t).$$

Here, $y(t)$ represents the position of the object, so that $y'(t)$ is its velocity and $y''(t)$ is its acceleration.

The constant $m > 0$ is the mass. The constant $\gamma \geq 0$ relates to damping. The constant $k > 0$ is the spring constant.

Note: The function $F(t)$ relates to external forces. We will assume that there is no external forcing, meaning that $F(t) = 0$.

Undamped Vibrations ($\gamma = 0$)

The equation of motion is $my'' + ky = 0$, or

$$y'' + \frac{k}{m}y = 0.$$

Let $\omega_0 = \sqrt{k/m}$ be the **natural frequency** and call $T = 2\pi/\omega_0$ the **period**.

The characteristic equation is $\lambda^2 + \omega_0^2 = 0$, which has roots $\lambda = \pm i\omega_0$. Thus, the general solution is:

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

By setting $c_1 = R\cos\delta$ and $c_2 = R\sin\delta$, we can write $y(t)$ in **phase-amplitude form**:

$$y(t) = R\cos(\omega_0 t - \delta).$$

Damped Vibrations ($\gamma > 0$)

The equation of motion is $my'' + \gamma y' + ky = 0$. The characteristic equation is $m\lambda^2 + \gamma\lambda + k = 0$, which has roots

$$\lambda_1, \lambda_2 = \frac{1}{2m}\left(-\gamma \pm \sqrt{\gamma^2 - 4mk}\right)$$

We have three cases:

(1) Under-damped motion: $\gamma^2 - 4mk < 0$. (Complex conjugate roots)
(2) Critically damped motion: $\gamma^2 - 4mk = 0$. (Real, repeated root)
(3) Over-damped motion: $\gamma^2 - 4mk > 0$. (Real, distinct roots)

2nd-Order Linear ODE: Non-homogeneous Equations

Goal: Solve 2nd-order linear ODE of the form

$$y'' + by' + cy = g(t).$$

The starting point will be the following:

Fact: The general solution to $y'' + by' + cy = g(t)$ is

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t),$$

where:
- ○ $c_1y_1(t) + c_2y_2(t)$ is the general solution to $y'' + by' + cy = 0$;
- ○ $y_p(t)$ is a particular solution to $y'' + by' + cy = g(t)$.

Note: This is just like the case of 1st-order linear systems $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$.

In that situation, the general solution was $\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \mathbf{x}_p(t)$, where $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ was a fundamental set for $\mathbf{x}' = A\mathbf{x}$.

Q: How to find the particular solution $y_p(t)$?
A: There are two methods:
- ○ Method of Undetermined Coefficients
- ○ Method of Variation of Parameters (for 2nd-order ODEs)

Method of Undetermined Coefficients

Goal: Solve 2nd-order linear ODE of the form

$$y'' + by' + cy = g(t).$$

Method (Undetermined Coefficients):

(1) Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous ODE $y'' + by' + cy = 0$.

(2) Make a "first guess" for the particular solution $y_p(t)$. (See Table.)

(3A) If the "first guess" has **no terms** which are combinations of $y_1(t), y_2(t)$, then that's your "actual guess" for $y_p(t)$.

(3B) If the "first guess" **does have** terms which are combinations of $y_1(t), y_2(t)$, then multiply your first guess by t or t^2 so that's no longer the case. The result is your "actual guess" for $y_p(t)$.

(4) Substitute your "actual guess" into the ODE. Solve for the undetermined coefficients. The general solution is $y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$.

$g(t)$	First Guess
e^{kt}	Ae^{kt}
$\cos(kt)$ or $\sin(kt)$	$A\cos(kt) + B\sin(kt)$
$P_n(t)$	$A_nt^n + \cdots + A_1t + A_0$
$P_n(t)e^{kt}$	$(A_nt^n + \cdots + A_1t + A_0)e^{kt}$
$P_n(t)\cos(kt)$ or $P_n(t)\sin(kt)$	$(A_nt^n + \cdots + A_0)\cos(kt)$ $+ (B_nt^n + \cdots + B_0)\sin(kt)$

Note: Here, $P_n(t)$ is a polynomial of degree n.

Example: Given $y'' = 12t^2$.

The corresponding homogeneous ODE is $y'' = 0$, which has general solution $c_1 + c_2t$. That is, $y_1(t) = 1$ and $y_2(t) = t$.

Since $g(t) = 12t^2$, our "first guess" is $y_p(t) = At^2 + Bt + C$. However, notice that both Bt and C are linear combinations of $y_1(t) = 1$ and $y_2(t) = t$. So, we modify our guess by multiplying by t^2. Our "actual guess" is

$$y_p(t) = t^2(At^2 + Bt + C) = At^4 + Bt^3 + Ct^2.$$

The general solution will end up being $y(t) = c_1 + c_2t + t^4$.