Review Problems

1. Consider the 1st-order ODE

\[ y' = (y + 1)(y - 3)(y - a). \]

(a) How many equilibrium solutions are there? (Your answer should depend on \( a \))
(b) Sketch solution curves (in the \( ty \)-plane) for different values of \( a \).
(c) Draw the bifurcation diagram.

2. Consider the IVP

\[ y' = 3x^2(y^2 + 1) \]
\[ y(0) = 0 \]

(a) Use the Improved Euler Method (with one step) to approximate \( y(0.1) \).
(b) Find the exact value of \( y(0.1) \).

3. Use Euler’s method (with one step) to approximate \( \ln(1.1) \).

4. Find the general solution of the 2nd-order ODE

\[ 2y'' - 12y' + 18y = 0. \]

5. Consider the 2nd-order IVP

\[ y'' + 9y = 0 \]
\[ y(0) = -2 \]
\[ y'(0) = 6. \]

(a) Find the solution.
(b) Find the smallest value of \( t > 0 \) for which \( y(t) \) is a local maximum.
(c) Graph the solution (in the \( ty \)-plane).

6. Consider the 2nd-order ODE

\[ y'' - (2a - 1)y' + a(a - 1)y = 0. \]

(a) Determine the values of \( a \), if any, for which all solutions tend to zero as \( t \to \infty \).
(b) Determine the values of \( a \), if any, for which all (non-zero) solutions become unbounded as \( t \to \infty \).
7. Consider the 1st-order IVP

\[(\cos t)y' + (\sin t)y = \sin t \cos t\]
\[y(\pi) = 0.\]

Find an interval on which the IVP will have exactly one solution.

8. Consider the 1st-order IVP

\[y' = \sqrt{4 - t^2 - y^2}\]
\[y(1) = 1.\]

Prove that there is some interval \((1 - h, 1 + h)\) on which the IVP has a unique solution \(y(t)\).

9. Consider the 2nd-order ODE

\[y'' - 5y' + 6y = 2e^t.\]

(a) Find the general solution using variation of parameters.
(b) Find the general solution using undetermined coefficients.

10. Consider the 2nd-order ODE

\[y'' + 2y' = 3 + 4 \sin 2t.\]

Find the general solution using undetermined coefficients.
Answers

1. (a) If $a \neq -1$ and $a \neq 3$, then there are three equilibrium solutions. If $a = -1$ or if $a = 3$, then there are only two equilibrium solutions.

2. (a) $y(0.1) = \frac{2}{3000} = 1.5 \times 10^{-3}$.
   
   (b) Separation of variables $\implies y(t) = \tan(x^3) \implies$ Exact value is $\tan\left(\frac{1}{1000}\right)$.

3. Use $y = \ln x$, so $y'(x) = 1/x$ and $y(1) = 0$. Answer is $\ln(1.1) \approx 1/10$.

4. $y(t) = c_1e^{3t} + c_2te^{3t}$.

5. (a) $y(t) = -2\cos(3t) + 2\sin(3t)$.
   
   (b)-(c). Write $y(t)$ in phase-amplitude form to get $y(t) = 2\sqrt{2}\cos(3t - \frac{3\pi}{4})$. One can then graph this.

6. Characteristic equation $\lambda^2 - (2a - 1)\lambda + a(a - 1) = 0$ has roots $\lambda = a$, $\lambda = a - 1$, which are real and distinct, so

   $$y(t) = c_1e^{at} + c_2e^{(a-1)t}.$$  

   Answer to (a): $a < 0$.
   Answer to (b): $a > 1$.

7. Divide by $\cos t$ to get

   $$y' + (\tan t)y = \sin t.$$  

   Answer is the interval $(\pi/2, 3\pi/2)$.

8. Let $f(t, y) = \sqrt{4 - t^2 - y^2}$. Calculate $\partial f/\partial y$. See that they’re both continuous on the open disk $t^2 + y^2 < 4$ in the $ty$-plane. Notice that the initial value $(1, 1)$ lies in the interior of this disk.

9. In both (a) and (b), the general solution is the same:

   $$y(t) = c_1e^{2t} + c_2e^{3t} + e^t.$$  

10. The general solution is

   $$y(t) = (c_1 + c_2e^{-2t}) + (-\frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t + \frac{3}{2}t).$$