Problems: Tue 6/27

1. Sketch the graph of \( f(x) = 1 + 2 \cos x \).

2. Sketch the graph of \( g(x) = \frac{3x + 1}{x} \).
   
   **Hint:** \( \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \).

3. Sketch the graph of the following functions.
   (a) \( F(x) = |\sin x| \).
   (b) \( G(x) = \sin(|x|) \).

4. Find the domain of \( h(x) = \frac{\tan x}{2^x \log_3(x)} \).

Even & Odd Functions

**Def:** Let \( f(x) \) be a function.

We say \( f \) is **even** if: \( f(-x) = f(x) \). (Symmetry in y-axis)

We say \( f \) is **odd** if: \( f(-x) = -f(x) \). (Symmetry in the origin)

5. Determine whether the following polynomials are even, odd, or neither.
   (a) \( p(x) = x^5 + 2x^3 + 7x \)
   (b) \( q(x) = x^4 - x \)
   (c) \( r(x) = x^6 - 3x^2 + 1 \)

Do you see a pattern? How can you quickly tell whether a polynomial is even, odd, or neither?

6. Are there any functions that are **both** even and odd? If so, which ones?

7. If \( f \) and \( g \) are even functions, is \( f + g \) also even?
Problems: Discontinuities: Thu 6/29

1A. Sketch the graph of \( f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases} \)

1B. Sketch the graph of \( g(x) = \frac{x^2}{x} \).

2A. Sketch the graph of \( \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases} \)

2B. Sketch the graph of \( F(x) = \frac{|x|}{x} \).

3A. Sketch the graph of \( H(x) = \begin{cases} \frac{1}{x^2} & \text{if } x < 0 \\ \sin(x) & \text{if } x \geq 0. \end{cases} \)

3B. Sketch the graph of \( K(x) = \sin \left( \frac{1}{x} \right) \).

Problems: Continuity

4. (a) Show that \( f(x) = 2^x(x^3 - 5) \) is continuous on \((-\infty, \infty)\).

(b) Show that \( g(x) = \frac{e^x}{\sin x} \) is continuous at \( x = \frac{\pi}{2} \).

(c) Show that \( h(x) = \cos(\ln x) \) is continuous on \((0, \infty)\).

5. Prove that the equation \( 2x + e^x = 3 \) has a solution in the interval \((0, 1)\).
1. Let \( f(x) = \frac{x - 2}{x^2 - 2x} \).

(a) Sketch the graph of \( f(x) \).

(b) Find \( f(2) \), if it exists.

(c) Find \( \lim_{x \to 2} f(x) \), if it exists.

(d) Is \( f(x) \) continuous at \( x = 2 \)? Give complete justification.

(e) Is \( f(x) \) continuous at \( x = 0 \)? Give complete justification.

(f) Is \( f(x) \) continuous at \( x = 3 \)? Give complete justification.
Problems: Wed 7/5

1. Evaluate \( \lim_{x \to 2\pi} \frac{x^3}{\cos x} \)

2. Evaluate \( \lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} \)

3. Evaluate \( \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t} \)

4. Show that \( f(x) = \begin{cases} x^2 + 2 & \text{if } x > 0 \\ 2 - x & \text{if } x \leq 0 \end{cases} \) is continuous at \( x = 0 \).

5. Evaluate \( \lim_{x \to 0} \frac{|x|}{x} \)

6. Evaluate \( \lim_{x \to 0} x^8 \arctan(x) \).

7. Evaluate \( \lim_{x \to 0} x^8 \arctan\left(\frac{1}{x}\right) \).
Problems: Thu 7/6

1. Show that \( f(x) = \frac{e^x}{\sin x} \) is continuous at \( x = \frac{\pi}{2} \).

2. Show that \( g(x) = 2^x(x^3 - 5) \) is continuous on \((-\infty, \infty)\).

3. Show that \( F(x) = \begin{cases} 
\sin(\pi x) & \text{if } x < 1 \\
\frac{2 - x}{2} & \text{if } x = 1 \\
\ln(x^2) & \text{if } x > 1
\end{cases} \) is continuous on \((-\infty, \infty)\).

4. Prove that the equation \( 2x + 3^x = 4 \) has a solution in the interval \((0, 1)\).
Problems: Vertical Asymptotes: Mon 7/10

1. Evaluate \( \lim_{x \to 1^-} \frac{x - 2}{(x - 1)^2} \) and \( \lim_{x \to 1^+} \frac{x - 2}{(x - 1)^2} \).

2. Evaluate \( \lim_{x \to 3^+} \ln(x^2 - 9) \).

3. Evaluate \( \lim_{x \to 2\pi^-} x \csc x \) and \( \lim_{x \to 2\pi^+} x \csc x \).

4. Find all vertical asymptotes of \( h(x) = \frac{x^3 - x}{x^2 - 6x + 5} \).

Problems: Horizontal Asymptotes

5. Evaluate \( \lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}} \).

6. Evaluate \( \lim_{x \to -\infty} \frac{x + 2}{\sqrt{9x^2 + 1}} \).

7. Evaluate \( \lim_{x \to \infty} \frac{\sin^2 x}{x^3} \).
Problems: Tue 7/11

1. Evaluate \( \lim_{x \to \pi^-} e^{\tan x} \) and \( \lim_{x \to \pi^+} e^{\tan x} \).

2. Find all vertical and horizontal asymptotes of \( h(x) = e^{\frac{3}{x-2}} \).

3. Evaluate \( \lim_{x \to \infty} \left( \sqrt{x^2 + 2x} - x \right) \).

4. Evaluate \( \lim_{x \to 0} x^4 e^{|\cos(1/x)|} \).

5. Let \( f(x) = \begin{cases} x \sin(1/x) & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \). Is \( f(x) \) continuous or discontinuous at \( x = 0 \)? Fully justify your answer.

6. Evaluate \( \lim_{x \to \infty} [\ln(2 + \sin x) - \ln(x)] \).
Problems: Wed 7/12

Problem 1: Find the equation of the tangent line to \( y = x^2 \) at \( x = 3 \).

Solution: Let \( f(x) = x^2 \). The point is \((3, f(3)) = (\quad)\). By the Point-Slope Formula, the tangent line is:

The slope of the tangent line at \( x = 3 \) is:

\[ m = f'(3) = \]

Problem 2: Find the equation of the tangent line to \( y = \sqrt{x} \) at \( x = 4 \).

Solution: Let \( f(x) = \sqrt{x} \). The point is \((4, f(4)) = (\quad)\). By the Point-Slope Formula, the tangent line is:

The slope of the tangent line at \( x = 4 \) is:

\[ m = f'(4) = \]
Problem 3: Prove that if \( f(x) \) and \( g(x) \) have derivatives, then

\[
\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).
\]

Solution: Use the definition of derivative:

\[
(f + g)'(x) = \lim_{h \to 0} \frac{(f + g)(x + h) - (f + g)(x)}{h}
\]

\[
= f'(x) + g'(x).
\]
Problems: Mon 7/17

In problems #1 to #5, find the derivative of the given function.

1. \( f(x) = x \cos(x) + \frac{3}{x^2} \).

2. \( g(x) = x^2 e^x \sec(x) \).

3. \( h(x) = 4 \cos(x) \tan(x) \).

\[ \text{4E.} \quad q(x) = 3x^3 \left( \frac{1}{x^3} - x^5 \right) \tan(x) \]

4. \( r(x) = \cot x \frac{\cot x}{x^3 + 1} + \frac{5}{\sqrt[3]{x}} \)

5E. Let \( p(x) = x^\pi e^x + \frac{\sqrt{x}}{\sqrt[3]{x}} \).

Find the equation of the tangent line to \( y = p(x) \) at \( x = 1 \).

6. Suppose \( f(x) = e^x g(x) \), where \( g(0) = 2 \) and \( g'(0) = 5 \).

Find \( f'(0) \).
Problems: Tue 7/18

In all problems, find the derivative of the given function.

1. \( F(x) = (4x - x^2)^{100} \)

2. \( f(x) = \sqrt[3]{1 + \tan x} \)

3. \( f(x) = \frac{1}{(x^2 + 1)^4} \)

4. \( f(x) = \sqrt{\frac{x}{x^2 + 4}} \)

5. \( f(x) = \sin(\cos(\tan x)) \)

6E. \( r(x) = \sec(2x) \ln(\sin^2 x) \)

7E. \( g(x) = \log_3(\sec(10\pi x)) \)

8E. \( f(x) = \cos^{2017}(x \arctan x + 4\pi) \)
Problems: Wed 7/19

Set A: Review

0. (HW #2) Find the constant $k$ for which the function

$$f(x) = \begin{cases} 
0.5x & \text{if } 0 \leq x \leq 1 \\
\sin(kx) & \text{if } 1 < x \leq 5 
\end{cases}$$

is continuous on the interval $[0, 5]$.

8E. Let $f(x) = \cos^{2017}(x \arctan x + 4\pi)$. Find $f'(x)$.

Set B: New Problems

1. Let $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 0$ and $f'(1) = 1$. Find $h'(1)$.

2. Suppose $f(x)$ is one-to-one and $f(4) = 5$ and $f'(4) = 3$. Find $(f^{-1})'(5)$.

3. (a) Find the 50th derivative of $g(x) = 2^x$

   (b) Find the 99th derivative of $h(x) = \sin x$.

   (c) Find a formula for the $n$th derivative of $f(x) = x^{-1}$

4. Find the values of $x$ at which the curve $y = x^4 + 4x^3 - 8x^2 - 48x + 1$ has a horizontal tangent line.

Challenge. Suppose that $f(x)$ is a differentiable function satisfying three properties:

   (a) $f(a + b) = f(a)f(b)$ for all real numbers $a, b$.

   (b) $f(0) = 1$

   (c) $f'(0) = 17$.

Find the function $\frac{d}{dx} \ln(f(x))$. 
Problems: Thu 7/20

1. Let \( F(x) = \begin{cases} 2x^2 & \text{if } x < 0 \\ 3x & \text{if } x \geq 0. \end{cases} \)
(a) Show that \( F(x) \) is continuous on \((-\infty, \infty)\).
(b) Show that \( F(x) \) is not differentiable at \( x = 0 \).

2. Let \( f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \)
Show that \( f(x) \) is not differentiable at \( x = 0 \).

3. Let \( g(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \)
(a) Show that \( g(x) \) is differentiable for \( x \neq 0 \), and find \( g'(x) \) for \( x \neq 0 \).
(b) Is \( g(x) \) differentiable at \( x = 0 \)? If so, find \( g'(0) \).
(c) Is \( g(x) \) continuous at \( x = 0 \)?

For Fun

4. (a) Sketch the graph of a function which is continuous everywhere, but not differentiable at exactly two points.
(b) Sketch the graph of a function which is continuous everywhere, but not differentiable at infinitely many points.

4. (c) Are there functions which are continuous everywhere, but differentiable at no point?
   If you say Yes: Can you describe such a function?
   If you say No: Give a reason why. (i.e.: You must argue: If a function is continuous everywhere, then it must be differentiable somewhere.)

Challenge. Suppose that \(|f(x)| \leq x^2\) for all real numbers \( x \).
Show that \( f(x) \) is differentiable at \( x = 0 \), and find \( f'(0) \).
Problems: Mon 7/24

1. Sketch the solution sets of the following equations.
   (a) \( xy - x^2 = 0 \)
   (b) \( (y - x)^2 = 1 \)
   (c) \( \sin(x^2 + y^2) = 0 \)

2. In each of the following, regard \( y \) as an implicit function of \( x \). Find \( dy/dx \).
   (a) \( 4 \cos(x) \sin(y) = 1 \)
   (b) \( y = \ln(x^2 + y^2) \)
   (c) \( e^y = x - y \)

3. Suppose \( x^4 + y^4 = 16 \), and regard \( y \) as an implicit function of \( x \). Find the second derivative \( y'' \).

4. Consider the astroid \( x^{2/3} + y^{2/3} = 1 \). Draw the segment of the tangent line to the astroid at the point \((a, b)\) cut off by the \( x \)- and \( y \)-axes.
   (a) Find the endpoints of this line segment.
   (b) Show that this line segment has length 1, no matter what point \((a, b)\) on the astroid is chosen.
Problems: Tue 7/25

Set A

4. Consider the astroid \( x^{2/3} + y^{2/3} = 1 \). Draw the segment of the tangent line to the astroid at the point \((a, b)\) cut off by the \(x\)- and \(y\)-axes.
   (a) Find the endpoints of this line segment.
   (b) Show that this line segment has length 1, no matter what point \((a, b)\) on the astroid is chosen.

Set B

In each of the following problems, find \(y'\).

1. \( y = (\cos x)^x \)

2. \( y = (\tan x)^{1/x} \)

3. \( y = (2x + 1)^5(x^4 - 3)^6 \)

4. \( y = \frac{\sin^2 x \tan^4 x}{(x^2 + 1)^2} \)
Problems: Wed 7/26

1. (a) Approximate $\sqrt{17}$ by using linear approximation.
   (b) Is your approximation in (a) larger or smaller than the actual value?

2. Let $f(x) = e^{3x}$.
   (a) Find the linear approximation (linearization) of $f(x)$ at $x = 0$.
   Conclude that for $x \approx 0$, we have $e^{3x} \approx 1 + 3x$.
   (b) Find the quadratic approximation of $f(x)$ at $x = 0$.
   Conclude that for $x \approx 0$, we have $e^{3x} \approx 1 + 3x + \frac{9}{2}x^2$.

3. Consider the equation $\cos(x) + 10x = 2$.
   (a) Show that this equation has a solution $x \in (0, \frac{\pi}{2})$.
   (b) Approximate this solution by using linear approximation at $x = 0$. 
Problems: Thu 7/27

Find the intervals on which the given function is increasing/decreasing. Identify any turning points.

1. $f(x) = 2x^3 + 3x^2 - 36x$.

2. $g(x) = \frac{x^2}{x^2 + 3}$

3. $h(x) = x^2 \ln(x)$

4. $p(x) = \ln(x^4 + 27)$
Problems: Mon 7/31

In each problem:

(a) Find the intervals on which the given function is increasing/decreasing. Identify any turning points.

(b) Find all the local maxima and local minima.

1. \( f(x) = x^5 + x^4 - 3x^3 + 7 \)

2. \( g(x) = \sqrt{x} e^{-x} \)

3. \( h(x) = \sqrt{x} (x + \sqrt{x}) \)

4. \( p(x) = \sqrt{x^2 + 1} - x \)

Problems: Tue 8/1

In each problem, sketch the graph of the function. Follow these steps:

(a) Find the intervals of increase and decrease. Identify any turning points.
(b) Identify local maxima and minima.
(c) Find the intervals where the function is concave up/down. Identify any inflection points.
(d) Find the vertical and horizontal asymptotes, if any.
(e) Sketch the function.

1. \( f(x) = x^4 + 8x^3 + 200 \)

2. \( g(x) = 3x^{2/3} - x \)

3. \( h(x) = x + \cos x \)

4. \( p(x) = e^{\arctan(x)} \)
Problems: Wed 8/2

In each problem: Find the absolute maximum and absolute minimum values of the function on the given set.

1. \( f(x) = (x^2 - 1)^3 \) on \([-2, 3]\).

2. \( g(x) = \ln(x^2 + x + 1) \) on \([-1, 1]\).

3. \( h(x) = x\sqrt{4 - x^2} \) on its domain.

Problems: Thu 8/3

1. A (right circular) cylinder is inscribed in a sphere of radius 3. Find the largest possible volume of such a cylinder.

2. A cylindrical can without a top is made to contain 64 cm\(^3\) of liquid. Find the radius that will minimize the cost of the metal to make the can.

3. Find the points on the ellipse \( 4x^2 + y^2 = 4 \) that are farthest away from the point \((1, 0)\).

4. Find the area of the largest rectangle that can be inscribed in the ellipse \( 4x^2 + y^2 = 4 \).
Problems: Mon 8/7

Set A

1. \( \lim_{x \to 1} \frac{\ln x}{\sin(\pi x)} \)

2. \( \lim_{x \to \pi} \frac{\sin x}{\cos x - 1} \)

3. \( \lim_{x \to 0} \frac{\sin x - x}{x^3} \)

4. (a) \( \lim_{x \to \infty} \frac{x}{\arctan(x)} \)
   
   (b) \( \lim_{x \to 0} \frac{x}{\arctan(x)} \)

Set B

1. \( \lim_{x \to 0^+} x \ln x \)

2. (a) \( \lim_{x \to \infty} x^2 e^x \)
   
   (b) \( \lim_{x \to -\infty} x^2 e^x \)

3. \( \lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) \)
Problems: Tue 8/8

1. \( \lim_{x \to 0^+} (1 + \sin 4x)^\cot x \)

2. \( \lim_{x \to 0} x^{\tan x} \)

3. Let \( f(x) = \frac{\sin x}{x} \)
   (a) Find the vertical asymptotes of \( f \), if there are any.
   (b) Find the horizontal asymptotes of \( f \), if there are any.

4. Let \( g(x) = \sqrt{x} \ln x \).
   (a) Find the vertical asymptotes of \( g \), if there are any.
   (b) Find the horizontal asymptotes of \( g \), if there are any.
   (c) Sketch the graph of \( g \). Label any turning points and inflection points.
Problems: Wed 8/9

1. A snowball melts so that its surface area decreases at a rate of 1 cm$^2$/min. Find the rate at which the radius decreases when the radius is 10 cm.

2. A triangle is changing in size. Its altitude is increasing at a rate of 1 cm/min, while its area is increasing at a rate of 2 cm$^2$/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm$^2$?

3. At noon, Ship A is 150 km west of Ship B. Suppose that Ship A is sailing east at 35 km/hr, and Ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4 pm?

Problems: Thu 8/10

1. The top of a ladder slides down a vertical wall at a rate of 0.15 m/sec. At the moment when the bottom of the ladder is 3 meters from the wall, it slides away from the wall at a rate of 0.2 m/sec. How long is the ladder?

2. Water is leaking out of an inverted conical tank at a rate of 1000 cm$^3$/min. At the same time, water is being pumped into the tank at a constant rate. The tank has height 6 meters, and the diameter at the top is 4 meters. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 meters, find the rate at which water is being pumped into the tank.

3. A trough is 10 feet long and its ends have the shape of isosceles triangles that are 3 feet across at the top and have a height of 1 foot. If the trough is being filled with water at a rate of 12 ft$^3$/min, how fast is the water level rising when the water is 6 inches deep?