The Cartan-Janet Theorem: Local Isometric Embedding of Real-Analytic Riemannian 2-Manifolds into $\mathbb{R}^3$

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Introduction

The goal of these notes is to prove:

Cartan-Janet (1927): Let (M^n, g) be a real-analytic Riemannian manifold, $N = \frac{1}{2}n(n+1)$. Every point of M has a neighborhood which has a real-analytic isometric embedding into $\mathbb{R}^N$.

We will prove the Cartan-Janet Theorem in the case $n = 2$, $N = 3$. The general case is almost exactly analogous to this one, but is much more notationally cumbersome, and requires an additional algebraic lemma.

To place this theorem in a broader context, we compare and contrast it with the better-known Nash Embedding Theorem, a global result:

Nash Embedding (1956): Let (M^n, g) be a C^k Riemannian manifold, where $3 \leq k \leq \infty$ or $k = \omega$. Then there exists a global C^k isometric embedding of M into some $\mathbb{R}^N$, where $N \leq \frac{1}{2}n(3n+11)$ if M compact, and $N \leq \frac{1}{2}n(3n^2 + 7n + 11) + (2n+1)$ if M is non-compact.

Note: As far as I know, local C^∞ embedding of M^n into $\mathbb{R}^N$ with $N = \frac{1}{2}n(n+1)$ is open!

Remark on the Isometric Embedding PDE

Let $\mathbf{u}\colon M^n \to \mathbb{R}^N$. Saying that $\mathbf{u}$ is an isometric embedding amounts to a system of fully nonlinear 1st-order PDE for $\mathbf{u}$. Namely, if $(x^1, \ldots, x^n)$ are local coordinates for M and $g = g_{ij}\, dx^i \circ dx^j$, then $\mathbf{u}$ is an isometric embedding if and only if

$$g_{ij} = \frac{\partial \mathbf{u}}{\partial x^i} \cdot \frac{\partial \mathbf{u}}{\partial x^j}.$$

This is a system of $\frac{1}{2}n(n+1)$ equations for the N unknown components of $\mathbf{u} = (u^1, \ldots, u^N)$.

EDS Basics

Def: An *exterior differential system* (EDS) is a pair $(M, \mathcal{I})$, where M is a smooth manifold, and $\mathcal{I} \subset \Omega^*(M)$ is a *differential ideal*.

A *differential ideal* is a graded ideal of $\Omega^*(M)$ that is closed under exterior differentiation.

Typically, differential ideals are described in terms of generators.

Def: Let $\phi_1, \ldots, \phi_s \in \Omega^*(M)$.

The *algebraic ideal generated by* $\phi_1, \ldots, \phi_s$ is defined by

$$\{\phi_1, \ldots, \phi_s\}_{\text{alg}} := \{\alpha_1 \wedge \phi_1 + \cdots + \alpha_s \wedge \phi_s \colon \alpha_i \in \Omega^*(M)\}$$

The *differential ideal generated by* $\phi_1, \ldots, \phi_s$ is defined by

$$\begin{aligned}\langle \phi_1, \ldots, \phi_s \rangle &:= \{\phi_1, \ldots, \phi_s, d\phi_1, \ldots, d\phi_s\}_{\text{alg}} \\ &= \{\alpha_1 \wedge \phi_1 + \cdots + \alpha_s \wedge \phi_s + \beta_1 \wedge d\phi_1 + \cdots + \beta_s \wedge d\phi_s \colon \alpha_i, \beta_i \in \Omega^*(M)\}\end{aligned}$$

Def: Let $(M, \mathcal{I})$ be an EDS.

An *integral manifold* of $(M, \mathcal{I})$ is a submanifold $F \colon N \to M$ such that $F^*\phi = 0 \ \forall \phi \in \mathcal{I}$.

Why might EDS arise in differential geometry? Essentially, if one approaches a problem with the language of moving frames, then relevant geometric quantities can typically be encoded as differential forms.

Remark: Nearly every (system of) PDE can be encoded as an EDS (with independence condition) in such a way that integral manifolds of the EDS are locally the (jet-)graphs of solutions to the PDE. In fact, there may be many ways to encode a PDE as an EDS. Heuristically, the space of EDS contains the space of PDE, and the problem of finding integral manifolds is much more general than that of finding local solutions to a system of PDE.

The central problem of the theory of EDS is to investigate the existence of integral manifolds. In analogy with how the Frobenius Theorem determines the existence of integral manifolds from a condition on tangent planes, we too wish to determine which "infinitesimal integral manifolds" can be sewn together to a *bona fide* integral manifold. Thus, we require a notion of "infinitesimal integral manifold":

Def: Let $(M, \mathcal{I})$ be an EDS.

A k-dim *integral element of* $\mathcal{I}$ is a k-plane $E \subset T_xM$ such that $\phi|_E = 0$ for all $\phi \in \mathcal{I}$.

We let $V_k(\mathcal{I}) \subset G_k(TM)$ denote the space of k-dim integral elements of $\mathcal{I}$.

That this definition captures the idea of "infinitesimal integral manifold" is explained by:

Fact: Let $F \colon N \to M$ be a submanifold. Then $F \colon N \to M$ is an integral manifold of $\mathcal{I}$ iff each $F_*(T_xN)$ is an integral element of $\mathcal{I}$.

Remark: The set $V_k(\mathcal{I}) \subset G_k(TM)$ is a closed subset. In the real-analytic case, it is an analytic variety. It can be very singular, and different strata may have different dimensions.

Note also that at any particular point of M, there many be no integral elements, exactly one, or an entire family of integral elements. Also, simply having a single integral element at every point is not in itself sufficient to guarantee the existence of integral manifolds.

One of the main tools for finding integral manifolds is the Cartan-Kähler Theorem. This is a geometric generalization of the Cauchy-Kovalevskaya Theorem in PDE, which is why real-analyticity is required. The idea is to build integral manifolds "successively," proceeding "one dimension at a time." That is, having found an integral 1-manifold, we thicken it to an integral 2-manifold, etc.

From this perspective, we see that what we really care about is not individual integral elements so much as nested sequences of integral elements. This leads to the following definition:

Def: Let $(M, \mathcal{I})$ be an EDS.

An *integral flag* at $x \in M$ is a sequence of subspaces $(0) \subset E_1 \subset \cdots \subset E_n \subset T_xM$ with $\dim(E_k) = k$ such that E_n is an integral element of $\mathcal{I}$ (so all E_k are integral elements of $\mathcal{I}$).

Which integral flags give rise to integral manifolds? The next definition will enable us to provide a sufficient condition that can be checked purely algebraically:

Def: Let $E \in V_k(\mathcal{I})$ be an integral element. Let $\{e_1, \ldots, e_k\}$ be a basis for $E \subset T_xM$.

The *space of polar equations of* E is the vector space

$$\mathcal{E}(E) := \{(e_1 \wedge \cdots \wedge e_k) \lrcorner \phi \mid \phi \in \mathcal{I} \cap \Omega^{k+1}(M)\} \subset T_x^*M.$$

Intuitively the 1-forms in the space of polar equations encode the "expected" conditions for a k-plane to be an integral element. The Cartan-Kähler Theorem says that when these expected conditions are realized, then integral manifolds exist:

Convention: From now on, all EDS under consideration will have no non-zero forms of degree 0.

Cartan-Kähler Theorem: Let $(M, \mathcal{I})$ be a real-analytic EDS (where $\mathcal{I}$ contains no non-zero forms of degree 0). Let $0 \subset E_1 \subset \cdots \subset E_n \subset T_xM$ be an integral flag of $\mathcal{I}$.

If in a neighborhood of E_n, the set $V_n(\mathcal{I})$ is a smooth manifold of codimension

$$\mathrm{codim}[V_n(\mathcal{I}) \subset G_n(TM)] = \dim \mathcal{E}(0) + \cdots + \dim \mathcal{E}(E_{n-1}),$$

then there exists a real-analytic integral n-manifold of $\mathcal{I}$ through x whose tangent space at x is E_n.

Def: For the purposes of this talk, we will make the following non-standard definition:

An integral flag $0 \subset E_1 \subset \cdots \subset E_n \subset T_xM$ is *ordinary* E_n has a neighborhood on which $V_n(\mathcal{I})$ is a smooth manifold and $\mathrm{codim}[V_n(\mathcal{I}) \subset G_n(TM)] = \dim \mathcal{E}(0) + \cdots + \dim \mathcal{E}(E_{n-1})$.

Strategy for Local Isometric Embedding

Phase 1: Define an EDS $\mathcal{I}_0$

∘ Prop 1: Locally: {Integral manifolds of $\mathcal{I}_0$} $\leftrightarrow$ {Isometric embeddings}

Phase 2: Define a larger EDS $\mathcal{I} \supset \mathcal{I}_0$

∘ Prop 2: Integral manifolds of $\mathcal{I}$ = Integral manifolds of $\mathcal{I}_0$

Phase 3: Find algebraic generators of $\mathcal{I}$ and determine certain compatibility conditions.

Phase 4: Use Cartan-Kähler to show: There exist integral manifolds of $\mathcal{I}$.

Phase 1: First Attempt at Defining the EDS

Setup: Fix a Riemannian 2-manifold (M^2, g). Let K denote its Gaussian curvature.

We fix a local orthonormal framing $\{E_1, E_2\}$ on M, with dual coframing $\{\eta_1, \eta_2\}$. Since our results will be local, we can assume that this coframing is defined on all of M.

Recall: By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form $\eta_{12} = -\eta_{21} \in \Omega^1(M)$ such that Cartan's First Structure Equations hold:

$$\begin{aligned} d\eta_1 &= \eta_{21} \wedge \eta_2 \\ d\eta_2 &= \eta_{12} \wedge \eta_1. \end{aligned}$$

Moreover, Cartan's Second Structure Equation also holds:

$$d\eta_{12} = K\,\eta_1 \wedge \eta_2.$$

Def: Let $\mathcal{F}_2(\mathbb{R}^3) \to \mathbb{R}^3$ denote the bundle over $\mathbb{R}^3$ whose elements consist of triples $(x; e_1, e_2)$, where $x \in \mathbb{R}^3$ and (e_1, e_2) is an orthonormal set of vectors in $\mathbb{R}^3$.

Note that $\mathcal{F}_2(\mathbb{R}^3)$ is diffeomorphic to $\mathbb{R}^3 \times \mathrm{SO}(3)$.

Let $U \subset \mathcal{F}_2(\mathbb{R}^3)$ be an open set on which there exists a real-analytic function $e_3 \colon U \to \mathbb{R}^3$ with the property that: for all $f = (x; e_1, e_2) \in U$, the vectors $\{e_1, e_2, e_3(f)\}$ form an orthonormal basis of $\mathbb{R}^3$.

We will regard the components of $f \in U$ as vector-valued functions $x, e_1, e_2 \colon U \to \mathbb{R}^3$. In so doing, we can define a set of 1-forms $\omega_i, \omega_{ij} \in \Omega^1(U)$ on $U \subset \mathcal{F}_2(\mathbb{R}^3)$ via

$$\begin{aligned} \omega_i &:= e_i \cdot dx \quad \text{for } 1 \leq i \leq 3, \\ \omega_{ij} &:= e_i \cdot de_j = -\omega_{ji} \quad \text{for } 1 \leq i < j \leq 3. \end{aligned}$$

Note that of these nine 1-forms, the set $\{\omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{31}, \omega_{32}\}$ gives a coframing of U.

Consider the EDS $(M^2 \times U^6, \mathcal{I}_0)$, where $\mathcal{I}_0$ is the differential ideal generated by the 1-forms

$$\mathcal{I}_0 = \langle \omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3 \rangle.$$

Convention: From now on, we adopt the index ranges $1 \leq i, j, k \leq 2$.

Prop 1: If $u\colon M \to \mathbb{R}^3$ is a local isometric embedding, then the graph of its lift $\widetilde{u}\colon M \to U$ is an integral 2-manifold of $\mathcal{I}_0$ on which $\omega_1 \wedge \omega_2$ is non-vanishing.

Conversely, every integral 2-manifold of $\mathcal{I}_0$ on which $\omega_1 \wedge \omega_2$ is non-vanishing is locally the graph of a function $F\colon M \to U$ for which $x \circ F\colon M \to \mathbb{R}^3$ is a local isometric embedding.

Proof: ($\Longrightarrow$) Let $u\colon M \to \mathbb{R}^3$ be an isometric embedding. Let $\widetilde{u}\colon M \to U$ be the lift of u – i.e.: $\widetilde{u}(z) := (u(z); du(E_1|_z), du(E_2|_z))$.

$$\begin{tikzcd} & U \subset \mathcal{F}_2(\mathbb{R}^3) \arrow[d, "x"] \\ M \arrow[ur, "\widetilde{u}"] \arrow[r, "u"'] & \mathbb{R}^3 \end{tikzcd}$$

Let $\Gamma_{\widetilde{u}} := \{(z, \widetilde{u}(z))\colon z \in M\} \subset M \times U$. Let's check that $\Gamma_{\widetilde{u}}$ is an integral manifold of $\mathcal{I}_0$ on which $\omega_1 \wedge \omega_2$ is non-vanishing.

$\circ$ Since $e_3(\widetilde{u}(z))$ is normal to $du(E_i|_z)$, we have

$$\widetilde{u}^*(\omega_3) = \widetilde{u}^* e_3 \cdot \widetilde{u}^*(dx) = (e_3 \circ \widetilde{u}) \cdot d(x \circ \widetilde{u}) = (e_3 \circ \widetilde{u}) \cdot du = 0.$$

$\circ$ Since u is an isometric embedding, we have, for all $v \in T_zM$,

$$\widetilde{u}^*(\omega_i)(v) = \widetilde{u}^* e_i \cdot \widetilde{u}^*(dx)(v) = (e_i \circ \widetilde{u}) \cdot du(v) = du(E_i|_z) \cdot du(v) = E_i|_z \cdot v = \eta_i(v).$$

$\circ$ On $\Gamma_{\widetilde{u}}$, we have $\omega_1 \wedge \omega_2 = \eta_1 \wedge \eta_2$. Since $\eta_1 \wedge \eta_2$ is non-zero when projected onto the factor M, it follows that $\omega_1 \wedge \omega_2$ is non-zero on $\Gamma_{\widetilde{u}}$.

($\Longleftarrow$) Let $X \subset M \times U$ be an integral 2-manifold of $\mathcal{I}_0$ on which $\omega_1 \wedge \omega_2$ does not vanish. Let $\pi\colon X \to M$ denote projection onto the first factor.

Since $\omega_i = \eta_i$ on X, we see that $\eta_1 \wedge \eta_2$ does not vanish on X. Thus, $T_pX \cap \mathrm{Ker}(D\pi_p) = 0$, so that $D\pi_p\colon T_pX \to T_{\pi(p)}M$ is an isomorphism, so $\pi\colon X \to M$ is locally a diffeomorphism. Therefore, X is locally the graph of some function $F\colon M \to U$.

Let $u := x \circ F\colon M \to \mathbb{R}^3$. We claim that $du(E_i) = e_i \circ F$, from which it follows that u is locally an isometric embedding.

Using the hypothesis $F^*\omega_3 = 0$, we have

$$0 = F^*\omega_3 = F^*(e_3 \cdot dx) = (e_3 \circ F) \cdot du.$$

Thus, $e_3(F(z))$ is normal to $du(E_i|_z)$ for all $z \in M$, so that the vectors $du(E_i|_z) \in \mathbb{R}^3$ are linear combinations of only $\{e_1(F(z)), e_2(F(z))\}$.

Using the hypothesis $F^*\omega_i = \eta_i$, we have, for all $v \in T_zM$:

$$E_i|_z \cdot v = \eta_i(v) = F^*(\omega_i)(v) = F^*(e_i \cdot dx)(v) = F^*e_i \cdot (F^*dx)(v) = e_i(F(z)) \cdot du(v).$$

In particular, $du(E_j|_z) \cdot e_i(F(z)) = E_i|_z \cdot E_j|_z = \delta_{ij}$.

Thus, $du(E_i) = e_i \circ F$, meaning that $du|_z$ takes the orthonormal basis $\{E_1|_z, E_2|_z\}$ of T_zM to the orthonormal basis $\{e_1(F(z)), e_2(F(z))\}$ of $du(T_zM)$. It follows that locally, u is an isometric embedding. $\diamondsuit$

Phase 2: A Better EDS

Let us now consider the EDS $(M^2 \times U^6, \mathcal{I})$, where $\mathcal{I}$ is generated by the 1-forms

$$\mathcal{I} = \langle \omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3, \omega_{12} - \eta_{12} \rangle.$$

We shall need the following structure equations:

$$\begin{aligned} d\omega_1 &= -\omega_{12} \wedge \omega_2 - \omega_{13} \wedge \omega_3 \\ d\omega_2 &= -\omega_{21} \wedge \omega_1 - \omega_{23} \wedge \omega_3 \\ d\omega_3 &= -\omega_{31} \wedge \omega_1 - \omega_{32} \wedge \omega_2 \\ d\omega_{12} &= -\omega_{13} \wedge \omega_{32}. \end{aligned}$$

Prop 2: Any integral 2-manifold of $\mathcal{I}_0$ on which $\omega_1 \wedge \omega_2$ does not vanish is an integral 2-manifold of the larger system $\mathcal{I}$.

Proof: Let X be an integral 2-manifold of $\mathcal{I}_0$ on which $\omega_1 \wedge \omega_2$ does not vanish. On X, we compute:

$$\begin{aligned} 0 &= d(\omega_1 - \eta_1) = -(\omega_{12} - \eta_{12}) \wedge \eta_2 \\ 0 &= d(\omega_2 - \eta_2) = -(\omega_{21} - \eta_{21}) \wedge \eta_1. \end{aligned}$$

Since the forms $\{\eta_1, \eta_2\}$ are linearly independent on X, we see that the 1-form $\psi := \omega_{12} - \eta_{12}$ must vanish on X. Thus, X is an integral manifold of $\mathcal{I}$. $\Diamond$

Remark: The geometric meaning of Prop 2 is that the Levi-Civita connection of an abstract metric is the same as the connection induced by any isometric embedding.

Phase 3: Algebraic Properties of $\mathcal{I}$

Algebraic Generators of $\mathcal{I}$

We begin by finding algebraic generators for $\mathcal{I}$. By definition, we have

$$\mathcal{I} = \{\omega_i - \eta_i, \omega_3, \omega_{12} - \eta_{12}, d(\omega_i - \eta_i), d\omega_3, d(\omega_{12} - \eta_{12})\}_{\text{alg}}.$$

We compute that

$$\begin{aligned} d(\omega_i - \eta_i) &\equiv 0 \mod \mathcal{I} \\ d\omega_3 &\equiv -\omega_{31} \wedge \omega_1 - \omega_{32} \wedge \omega_2 \mod \mathcal{I} \\ d(\omega_{12} - \eta_{12}) &\equiv \omega_{31} \wedge \omega_{32} - K\,\omega_1 \wedge \omega_2 \mod \mathcal{I} \end{aligned}$$

Thus, $\mathcal{I}$ is generated algebraically as

$$\mathcal{I} = \{\omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3, \omega_{12} - \eta_{12}, \Phi, \Psi\}_{\text{alg}},$$

where

$$\begin{aligned} \Phi &:= \omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2 \\ \Psi &:= \omega_{31} \wedge \omega_{32} - K\,\omega_1 \wedge \omega_2. \end{aligned}$$

Note that these are exactly the forms which appear in the Gauss and Codazzi Equations.

Conditions on 2-dim Integral Elements of $\mathcal{I}$

Let $E \subset T_{(x,f)}(M \times U)$ be a 2-dimensional integral element of $\mathcal{I}$ on which $(\omega_1 \wedge \omega_2)|_E \neq 0$. Since $\{\omega_1, \omega_2\}$ is linearly independent on E, we can write

$$\omega_{3i} = \sum_j h_{ij}\,\omega_j$$

for some numbers $h_{ij} \in \mathbb{R}$ (which depend on E). We observe the following:

∘ The condition that $\Phi|_E = 0$ is the condition that

$$h_{12} = h_{21}.$$

Geometrically, this says that the second fundamental form of the embedding should be symmetric.

∘ The condition that $\Psi|_E = 0$ is the condition

$$h_{11}h_{22} - (h_{12})^2 = K.$$

Geometrically, this is the Gauss equation: after being embedded in $\mathbb{R}^3$, the Gaussian curvature K of the abstract Riemannian 2-manifold should equal the determinant of the shape operator.

Phase 4: Proof of Cartan-Janet

Cartan-Janet Theorem: If the Riemannian metric g on M^2 is real-analytic, then every point of M has a neighborhood which has a real-analytic isometric embedding into $\mathbb{R}^3$.

Proof: By Prop 1 and Prop 2, integral 2-manifolds of $(M \times U, \mathcal{I})$ on which $\omega_1 \wedge \omega_2$ is non-vanishing give rise to local isometric embeddings. Thus, the problem amounts to constructing integral 2-manifolds of $\mathcal{I}$.

By the Cartan-Kähler Theorem, if at $(x, f) \in M \times U$ there is a 2-dimensional integral flag $0 \subset E_1 \subset E_2 \subset T_{(x,f)}(M \times U)$ that is *ordinary*, then there exists an integral 2-manifold of $\mathcal{I}$ through (x, f) tangent to E_2.

To this end, we will construct an open submanifold $\Lambda(Z) \subset V_2(\mathcal{I}, \omega)$ such that:

- ∘ At every $(x, f) \in M \times U$, there exist integral elements in $\Lambda(Z)$ over (x, f)
- ∘ Every integral element in $\Lambda(Z)$ is the terminus of an ordinary integral flag.

Let $Z \subset M \times U \times \mathbb{R}^3$ denote the subset

$$Z = \{(x, f, h) \in M \times U \times \mathbb{R}^3 \colon h_{11}h_{22} - (h_{12})^2 = K(x) \text{ and } h_{11} \neq 0\}.$$

By the Implicit Function Theorem, Z is a smooth submanifold of $M \times U \times \mathbb{R}^3$ of codimension 1, so that

$$\dim(Z) = \dim(M^2 \times U^6 \times \mathbb{R}^3) - 1 = 10.$$

Define a map $\Lambda \colon Z \to V_2(\mathcal{I}, \omega)$ via

$$\begin{aligned} \Lambda(x, f, h) &:= \text{the 2-plane } E \in V_2(\mathcal{I}, \omega) \text{ at } (x, f) \text{ s.t. } \omega_{3i} = \sum h_{ij}\, \omega_j. \\ &= \operatorname{Ker}\Big\{\omega_i - \eta_i, \omega_3, \omega_{12} - \eta_{12}, \omega_{3i} - \sum h_{ij}\, \omega_j\Big\}. \end{aligned}$$

It is clear that Λ is smooth and injective. One can check that, in fact, Λ is a smooth embedding. Moreover, $\Lambda(Z) \subset V_2(\mathcal{I}, \omega)$ is an open subset.

Picture: $V_2(\mathcal{I}, \omega) \to M \times U$ is surjective, with fibers (real) quadric surfaces in $\mathbb{R}^3$.

We claim that every $E \in \Lambda(Z) \subset V_2(\mathcal{I}, \omega)$ is the terminus of an ordinary integral flag – i.e.: every $E \in \Lambda(Z)$ admits a filtration $0 \subset E_1 \subset E$ such that near E, the subset $V_2(\mathcal{I}, \omega) \subset G_2(T(M \times U))$ is a smooth manifold of codimension

$$\operatorname{codim}[V_2(\mathcal{I}, \omega) \subset G_2(T(M \times U))] = \dim \mathcal{E}(0) + \dim \mathcal{E}(E_1).$$

Indeed, $\Lambda(Z)$ is itself an open submanifold of $V_2(\mathcal{I}, \omega)$ containing E. That is, near E, we have that $V_2(\mathcal{I}, \omega)$ is a smooth manifold of codimension

$$\operatorname{codim}[V_2(\mathcal{I}, \omega) \subset G_2(T(M \times U))] = 20 - 10 = 10.$$

On the other hand, let $E = \Lambda(x, f, h) \in \Lambda(Z)$, and let $E_1 \subset E$ denote the subspace annihilated by ω_2. We claim that $\dim \mathcal{E}(0) + \dim \mathcal{E}(E_1) = 10$.

Note that we have a coframing of $M \times U$ given by

$$\{\omega_1, \omega_2, \omega_3, \eta_1, \eta_2, \omega_{12}, \omega_{31}, \omega_{32}\}.$$

However, a coframing better suited to our problem is provided by

$$\{\omega_1, \omega_2, \omega_3, \omega_1 - \eta_1, \omega_2 - \eta_2, \omega_{12} - \eta_{12}, \pi_1, \pi_2\},$$

where

$$\pi_1 := \omega_{31} - \sum h_{1j}\, \omega_j \quad \text{and} \quad \pi_2 := \omega_{32} - \sum h_{2j}\, \omega_j.$$

Let's express the forms in $\mathcal{I}$ in terms of this coframing. Since

$$\mathcal{I} = \{\omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3, \omega_{12} - \eta_{12}, \Phi, \Psi\}_{\text{alg}},$$

we only need to re-express Φ and Ψ. A straightforward calculation gives

$$\begin{aligned} \Phi &= \pi_1 \wedge \omega_1 + \pi_2 \wedge \omega_2 \\ \Psi &= (h_{12}\pi_1 - h_{11}\pi_2) \wedge \omega_1 + (h_{22}\pi_1 - h_{12}\pi_2) \wedge \omega_2 + \pi_1 \wedge \pi_2. \end{aligned}$$

From this, it is easy to see that

$$\mathcal{E}(0) = \mathcal{I} \cap \Omega^1(M \times U) = \{\omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3, \omega_{12} - \eta_{12}\}$$

and

$$\begin{aligned} \mathcal{E}(E_1) &= \{e_1 \lrcorner\, \phi \mid \phi \in \Omega^2(M \times U)\} \\ &= \{\omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3, \omega_{12} - \eta_{12}, e_1 \lrcorner\, \Phi, e_1 \lrcorner\, \Psi\} \\ &= \{\omega_1 - \eta_1, \omega_2 - \eta_2, \omega_3, \omega_{12} - \eta_{12}, \pi_1, h_{12}\pi_1 - h_{11}\pi_2\}. \end{aligned}$$

Therefore,

$$\dim \mathcal{E}(0) + \dim \mathcal{E}(E_1) = 4 + 6 = 10.$$

Thus, $0 \subset E_1 \subset E$ is an ordinary integral flag.

By Cartan-Kähler, there exist integral 2-manifolds tangent to E on which $\omega_1 \wedge \omega_2$ is non-vanishing. By Prop 1 and Prop 2, this implies the existence of local isometric embeddings. ♠

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