Problem Set VIII — Math 53h

Due **Tuesday**, June 2, 5pm, to office 380-383Z

Exercise 1: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map; let Γ be the periodic orbit generated by a periodic solution $\mathbf{x}(t)$ of $\mathbf{x}' = F(\mathbf{x})$. Explain the construction of a Poincare map of this orbit.

Exercise 2: Consider an $\mathbb{R}^2$ autonomous system expressed in polar coordinates:

$$r' = r(r-1)(r-2), \quad \theta' = 1;$$

(1) find closed orbit and equilibrium point of this system;
(2) indicate the limiting behavior of all other non-constant solutions.

Exercise 3: Consider the system

$$x' = y - x^3 + 8x, \quad y' = -x.$$

(1) Sketch the argument showing that there is a Poincare map $P : v^+ \to v^+$ using the flow of this system, where v^+ is the positive part of the y-axis.
(2) Show in details that there is one and only on fixed point $\bar{\alpha}$ of P, and for any $\alpha \in (0, \infty)$, $\lim_{n\to\infty} P^n(\alpha) = \bar{\alpha}$.
(3) Indicate the implication of (2) to the limiting behavior of solutions of this system.
(Hint: A similar system is studied in details in [page 262-270, Hirsch].)

Exercise 4: [Page 65, Brendle] Problem 5.1.

Exercise 5: [Page 65, Brendle] problem 5.3.

Exercise 6: (Complete the proof of Poincare-Bendixson Theorem) Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ a C^1-map, and let Ω be the ω-limit set of a solution $\mathbf{x}(t)$, which we know is non-empty, bounded, and contains no equilibrium point of $\mathbf{x}' = F(\mathbf{x})$. Suppose Ω contains the orbit Γ of a non-constant periodic solution $\mathbf{x}(t)$. Show that $\Omega = \Gamma$.
(Hint: Using that Ω is connected, thus if $\Omega \neq \Gamma$, then there is a sequence $z_n \in \Omega - \Gamma$ so that $z_n \to \bar{z} \in \Gamma$. Then pick a transversal line segment S passing through $\bar{z}$. Argue that $S \cap \Omega$ contains more than one point, violating that $\#(\Omega \cap S) \leq 1$.)