Problem Set VII — Math 53h

Due **Wednesday**, May 20, 5pm, to office 380-383Z

Exercise 1: [Brendle] p65, problem 5.1.

Exercise 2: [Brendle] p67, problem 5.5.

Exercise 3: Let $f(x, y) = x^2 + y^4 + a$. Study the system

$$x' = -f_y - f \cdot f_x, \quad y' = f_x - f \cdot f_y.$$

(1). For $a = -1$, study its equilibrium(s), it ω-limit set(s), and its closed orbit(s), if any. Also sketch a few orbits to indicate the qualitative properties of this systems
(2). Do the same thing for $a = 1$.

Exercise 4: Let F be a smooth vector field on a neighborhood of the annular region $A = \{1 \leq x^2 + y^2 \leq 2\}$. Suppose for every boundary point $p \in \partial A$, $F(p)$ is a non-zero vector tangent to the boundary ∂A.
(1) Sketch the possible phase portraits in A under the further assumption that there are no equilibria and no closed orbits besides the boundary circles. (Consider separately the cases where both flows on the boundaries are counterclosewise or travel oppositely.)
(2). In case both flows on the two boundaries are counterclosewise, is it possible that there is an hyperbolic equilibrium in A? Of the type sink, source or saddle? Draw possible phase portraits.
(3). In case the two flows on the boundaries travels oppositely, is it possible that there is an hyperbolic equilibrium in A? Of the type sink, source or saddle? Draw possible phase portraits.

Exercise 5: The system

$$x' = x(2 - x - y), \quad y' = y(3 - 2x - y)$$

is a model of competing species. Determine its phase portraits; explain why these equations make it mathematically possible, but extremely unlikely, for both species to survise.

Exercise 6: A modification of the predator/prey equations is given by

$$x' = x(1-x) - \frac{axy}{x+1}, \quad y' = y(1-y); \qquad a > 0.$$

(1) Find all equilibrium points and classify them.
(2) Sketch the null clines and the phase portraits for different values of a.
(3) Describe any bifurcations that occur as a varies.

Exercise 7: Consider the system

$$x' = x(1-x) - \frac{axy}{x+b}, \quad y' = y(1 - \frac{y}{x}), \qquad a,\, b > 0.$$

Determine the region in the parameter space (a, b) for which the system has a stable equilibrium with both x, $y \neq 0$. Prove that, if the equilibrium points are unstable, this system has a stable limit cycle.

Exercise 8:(Optional) Indicate how to construct a 3-dimensional system so that there is an ω-limit set that is two dimensional.
(Hint: First, use polar coordinates to help construct an $\mathbb{R}^2$-system in xy-plane so that there is an ω-limit set that is a closed orbit. Then use translation to make it to locate in $x > 0$. Then use cylindrical coordinates (x, y, φ) to build a system by adding an equation $\varphi' = \alpha$ for an appropriately chosen α.)