Problem Set VI — Math 53h

Due **Tuesday**, May 12, 5pm, to office 380-383Z

Exercise 1: Sketch the phase portraits for the following systems. Determine if the system is Hamiltonian or gradient along the way.
(a). $x' = x + 2y$, $y' = -y$.
(b). $x' = x^2 - 2xy$, $y' = y^2 - x^2$.
(c). $x' = -\sin^2 x \sin y$, $y' = -2\sin x \cos x \cos y$.

Exercise 2: We consider the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x - y^2 \\ -y + x^2 \end{bmatrix}$$

Its equilibrium 0 is hyperbolic. Suppose for sufficiently small $\epsilon > 0$, and any real α, we let $\begin{bmatrix} x(t)_\alpha \\ y(t)_\alpha \end{bmatrix}$ be the solution of the above system with initial condition $x(0)_\alpha = \alpha$ and $y(0)_\alpha = \epsilon$.
Suppose we know for two reals $a < b$, both $\begin{bmatrix} x(t)_a \\ y(t)_a \end{bmatrix}$ and $\begin{bmatrix} x(t)_b \\ y(t)_b \end{bmatrix}$ exist for all time $t \geq 0$, and satisfy

$$-y(t)_a \leq 4\,x(t)_a \leq y(t)_a \text{ and } -y(t)_b \leq 4\,x(t)_b \leq y(t)_b, \quad \forall t \geq 0.$$

(1). Show that for all $c \in (a, b)$, the solution $\begin{bmatrix} x(t)_c \\ y(t)_c \end{bmatrix}$ exits for all time $t \geq 0$, and also has $-y(t)_c \leq 4\,x(t)_c \leq y(t)_c$ for all $t \geq 0$.
(2). For any $c \in [a, b]$, we let $f_c(t) = \frac{1}{1 - \frac{x(t)_c^2}{y(t)_c}}$. Show that one can solve $g_c(s) : [0, \infty) \to [0, \infty)$ so that

$$g_c(s)' = f_c(g_c(s)).$$

Show that $g_c(s)$ is strictly increasing and $\lim_{s \to \infty} g_c(s) = \infty$. Also $g_c(s)$ is C^1 in (c, t).
(3). Show that

$$y(g_c(s))_c = \epsilon e^{-s}.$$

Calculate

$$\frac{\partial}{\partial c}\left(\frac{d}{ds}\left(x(g_c(s))_c\right)\right).$$

Can you see that it is always positive?
(Hint: (1). Find a bound of $x(t)_c'$ and $y(t)_c'$. (2). Argue that $f_c(t)$ is smooth for

all $t \geq 0$ by quoting a smoothness results. Here the bound in (1) is needed. (3). use the chain rule and the equation; the expression involves $\frac{\partial}{\partial c} x(g_c(s))_c$. Argue that $\frac{\partial}{\partial c} x(g_c(s))_c \geq 0$. **Remark**: This is to fill in the details of the sketchy part of the proof of the existence of stable manifolds in 2-dimensional case presented in the book [p.169-171, HSD]. From (3), one sees that we have $\lim_{t\to\infty} y(t)_c = 0$ for $c \in [a, b]$.)

Exercise 3: Study in details the following equation

$$x'' + \omega^2 x + bx^3 = 0, \quad \omega \neq 0, b \in \mathbb{R}. \tag{1}$$

(1). Convert it into a linear system; show that it is Hamiltonian; find $H(x, y)$.
(2). Show that for small a (i.e. $|a|$ is small), the solution $x_a(t)$ to the above equation with $x_a(0) = a$ and $x_a(0)' = 0$ exists for all time $t \in \mathbb{R}$ and is periodic.
(3). Find an integral expression of the period T_a of the solution $x_a(t)$.
(4). Expand of $x_a(t)$ in power series in a:

$$x_a(t) = x(t)_0 + x(t)_1 a + x(t)_2 a^2 + x(t)_3 a^3 + O(a^4),$$

and derive equations satisfied by $x(t)_k$ for $k = 0, \cdots, 3$; derive the initial conditions $x(0)_k$ and $x(0)'_k$ for $k = 0, \cdots, 3$.
(5). Solve $x(t)_k$ explicitly, for $k = 0, \cdots, 3$.
(6). Expand $T_a = c_0 + c_1 a + c_2 a^2 + O(a^3)$, and use $x_a(T_a) = a$ and/or $x_a(T_a)' = 0$ to determine c_0, c_1 and c_2.
(Hint: (2) show the H has a local minimum at $(0, 0)$, and the level curves are smooth near this point. (3) use the method in the textbook. (4) and etc. plug $x_a(t)$ into (1), and compare coefficients of a^k.)