Problem Set V — Math 53h

Due **Wednesday**, May 6, 5pm, to office 380-383Z

Exercise 1: For each of the following nonlinear systems,
(a) find all of the equilibrium points and describe the behavior of the associated liberalized system.
(b) describe their phase portraits. If there is a stable manifold, find it.
(c) does the linearized system accurately describe the local behavior near the equilibrium points?

1. $x' = x + y^2, \quad y' = -2y$
2. $x' = x^2, \quad y' = y^2$
3. $x' = x(x^2 + y^2), \quad y' = y(x^2 + y^2)$.

Exercise 2: Consider the system

$$x' = y - x^5, \quad y' = -x - y^3.$$

Use the function $L(x, y) = x^2 + y^2$ to show that 0 is asymptotically stable. (Thus L is a Lyapunov function.)

Exercise 3: Consider the system

$$\begin{aligned} x'(t) &= x(t) - \sqrt{x(t)^2 + y(t)^2}\,(x(t) + y(t)) + x(t)y(t) \\ y'(t) &= y(t) + \sqrt{x(t)^2 + y(t)^2}\,(x(t) - y(t)) - x(t)^2. \end{aligned}$$

(i) Show that in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), the system can be rewritten as

$$r'(t) = r(t)\,(1 - r(t)), \qquad \theta'(t) = r(t)\,(1 - \cos\theta).$$

(ii) Show that if $(x(0), y(0)) \neq 0$, then $\lim_{t\to\infty}(x(t), y(t)) = (1, 0)$.
(iii) Is $(1, 0)$ a stable equilibrium point?

Exercise 4: Let $F : U \to \mathbb{R}^n$ be C^1, $\Phi(x, t)$ is the flow of $x' = F(x)$, namely $\Phi(x, t)' = F(\Phi(x, t))$. (In the text book, $\Phi(x, t)$ is denoted by $\varphi_t(x)$.) Suppose $x_0 \in U$ and $T > 0$ such that $\Phi(x_0, T)$ is defined. Let

$$C = \{\Phi(x_0, t) \mid t \in [0, T]\} \subset U \quad \text{and} \quad C_r = \{x \in \mathbb{R}^n \mid \text{dist}(x, C) \leq r\}.$$

We choose $r > 0$ so that $C_{4r} \subset U$. We let

$$L = \sup_{x \in C_{4r}} \| DF(x) \|_{op} .$$

(1) Suppose $t_0 \in [0, T]$ so that $\Phi(y_0, t_0)$ is defined and $\| \Phi(y_0, t) - \Phi(x_0, t) \| \leq 4r$ for all $t \in [0, t_0]$. Show that

$$\| \Phi(y_0, t) - \Phi(x_0, t) \| < e^{Lt} \| y_0 - x_0 \|, \quad t \in [0, t_0].$$

(2). Show that for all $\| y_0 - x_0 \| < e^{-LT} r$, $\Phi(y_0, T)$ is defined and

$$\| \Phi(y_0, T) - \Phi(x_0, T) \| < e^{LT} \| y_0 - x_0 \| .$$

(Hint: For (1), if you have proved this is the previous problem set, just stated it and you can skip it. For (2), suppose $\Phi(y_0, T)$ is not defined, then there is a time $t_0 \in [0, T]$ where $\Phi(y_0, t_0)$ is defined and $\| \Phi(y_0, t_0) - \Phi(x_0, t_0) \| = 4r$; use (1) to derive a contradiction. If $\Phi(y_0, T)$ is defined and for some $t_0 \in [0, T]$, $\| \Phi(y_0, t_0) - \Phi(x_0, t_0) \| = 4r$, derive a contradiction. Ruling out the previous two situations, apply (1) to prove (2).)