Problem Set VI — Math 53h

Due **Thursday**, 4/30, 5pm, to office 380-383Z

Exercise 1: Let $U \subset \mathbb{R}^n$ be an open subset and $F : U \to \mathbb{R}^n$ be continuous. Let $x_0 \in U$. Choose $r > 0$ and M so that $\overline{B_r(x_0)} \subset U$ and $\| F(x) \| \leq M$ for every $x \in \overline{B_r(x_0)}$. We set $\delta = r/M$, and defined a sequence of functions $x_k(t) : [0, \delta] \to \mathbb{R}^n$ inductively by

$$\begin{cases} x_k(0) = x_0; \\ x_k(t) = x_k(\frac{j\delta}{k}) + (t - \frac{j\delta}{k})F(x_k(\frac{j\delta}{k})), \quad t \in [\frac{j\delta}{k}, \frac{(j+1)\delta}{k}]. \end{cases}$$

Prove that x_k are well-defined, and there is a subsequence $x_{k_n}(t)$ that is uniformly converges to a continuous function $x(t)$ on $[0, \delta]$.

Exercise 2: Let U be the unit disk in $\mathbb{R}^n$ centered at 0, and $F : U \to U$ is a continuous map so that $\|F(x) - F(y)\| \leq \frac{1}{2} \|x - y\|$. So that if we pick $x_0 \in U$, and inductively define $x_k = F(x_{k-1})$, then x_k converges to a point $\bar{x} \in U$ so that $F(\bar{x}) = \bar{x}$.

Exercise 3: Let $U \subset \mathbb{R}^n$ be open, $F : U \to \mathbb{R}^n$ a C^1 map, and $x_0 \in U$. Show that for sufficiently small $\delta > 0$, the $x_k(t) : [0, \delta] \to \mathbb{R}^n$ defined inductively by

$$x_1(t) = x_0, \quad x_{k+1}(t) = x_0 + \int_0^t F(x_k(s))ds,$$

are well-defined, and converge uniformly to an $x(t)$ such that

$$x(t) = x_0 + \int_0^t F(x(s))ds.$$

Exercise 4: Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^n$ be C^1. Explain the reason why the maximal domain $\Omega \subset U \times \mathbb{R}$ of the flow $\Phi(x_0, t)$ associated to the system $x' = F(x)$ is open. (Remark: Give a heuristic argument.)