Problem Set II — Math 53h

Due Tuesday, 4/14, 5pm, to office 380-383Z

Exercise 1: Let $A \in M_{n\times n}(\mathbb{C})$. Let λ be an eigenvalue of A of multiplicity v. Show that for any integer $k \geq v$,

$$\ker(A-\lambda I)^k = \ker(A-\lambda I)^v.$$

Exercise 2: For $A \in M_{n\times n}(\mathbb{C})$, we denote by $\mathrm{tr}(A)$ the trace of A.
(i) Show that for S invertible, $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(A)$.
(ii) Show that for every $n \times n$ matrix A, $\det(\exp(tA)) = e^{t\,\mathrm{tr}(A)}$.

Exercise 3: Let $A \in M_{n\times n}\mathbb{C}$ with distinguished eigenvalues $\lambda_1, \cdots, \lambda_m$. Let $f(\lambda) = (\lambda-\lambda_1)\cdots(\lambda-\lambda_m)$. Show that A is diagonalizable if and only if $f(A) = 0$.
Exercise 4: Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(i). Find its eigenvalues, their multiplicities, and their corresponding generalized eigenspaces.
(ii). Verify that $\mathbb{C}^4$ is a direct sum of the generalized eigenspaces of A.
(iii). Find L in the $A = L+N$ decomposition. Find S and a diagonal D so that $L = SDS^{-1}$.
(iv). Find N and verify that $LN = NL$ and $N^4 = 0$.
(v). Evaluate $\exp(At)$.
Exercise 5: Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

(i). Find its eigenvalues, their multiplicities, and their corresponding generalized eigenspaces.
(ii). Verify that $\mathbb{C}^4$ is a direct sum of the generalized eigenspaces of A.
(iii). Find L in the $A = L+N$ decomposition. Find S and a diagonal D so that $L = SDS^{-1}$.
(iv). Find N and verify that $LN = NL$ and $N^4 = 0$.
(v). Evaluate $\exp(At)$.

Hints:
1. Can use that for $k \geq v$, $g.c.d.\big((x-\lambda)^k, p_A(x)\big) = (x-\lambda)^v$.
2. For i, can use $p_A(x) = x^n - \mathrm{tr}(A)x^{n-1} + \cdots$. For (ii), can first show this identity for A diagonalizable; or use $L+N$ decomposition.
3. First show in case $A = L + N$ has L diagonal; or use that $g.c.d.\big((\lambda - \lambda_k)^{v_k}, f(\lambda)\big) = (\lambda - \lambda_k)$.