Exercise 1: Let $A \in M_{n \times n}(\mathbb{C})$. Let $\lambda$ be an eigenvalue of $A$ of multiplicity $v$. Show that for any integer $k \geq v$, \[ \ker(A - \lambda I)^k = \ker(A - \lambda I)^v. \]

Exercise 2: For $A \in M_{n \times n}(\mathbb{C})$, we denote by $\text{tr}(A)$ the trace of $A$.
(i) Show that for $S$ invertible, $\text{tr}(S^{-1}AS) = \text{tr}(A)$.
(ii) Show that for every $n \times n$ matrix $A$, $\det(\exp(tA)) = e^{t\text{tr}(A)}$.

Exercise 3: Let $A \in M_{n \times n}(\mathbb{C})$ with distinguished eigenvalues $\lambda_1, \cdots, \lambda_m$. Let $f(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_m)$. Show that $A$ is diagonalizable if and only if $f(A) = 0$.

Exercise 4: Let
\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -2 & 1 & -1 \\
0 & 0 & -2 & 1 \\
0 & 0 & 0 & -2
\end{bmatrix}
\]
(i). Find its eigenvalues, their multiplicities, and their corresponding generalized eigenspaces.
(ii). Verify that $\mathbb{C}^4$ is a direct sum of the generalized eigenspaces of $A$.
(iii). Find $L$ in the $A = L + N$ decomposition. Find $S$ and a diagonal $D$ so that $L = SDS^{-1}$.
(iv). Find $N$ and verify that $LN = NL$ and $N^4 = 0$.
(v). Evaluate $\exp(At)$.

Exercise 5: Let
\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]
(i). Find its eigenvalues, their multiplicities, and their corresponding generalized eigenspaces.
(ii). Verify that $\mathbb{C}^4$ is a direct sum of the generalized eigenspaces of $A$.
(iii). Find $L$ in the $A = L + N$ decomposition. Find $S$ and a diagonal $D$ so that $L = SDS^{-1}$.
(iv). Find $N$ and verify that $LN = NL$ and $N^4 = 0$.
(v). Evaluate $\exp(At)$.
Hints:
1. Can use that for $k \geq v$, $g.c.d.((x - \lambda)^k, p_A(x)) = (x - \lambda)^v$.
2. For i, can use $p_A(x) = x^n - \text{tr}(A)x^{n-1} + \cdots$. For (ii), can first show this identity for $A$ diagonalizable; or use $L + N$ decomposition.
3. First show in case $A = L + N$ has $L$ diagonal; or use that $g.c.d.((\lambda - \lambda_k)^v, f(\lambda)) = (\lambda - \lambda_k)$.