Exercise 1. Find the general solution of the logistic differential equation with constant harvesting,

\[ x' = x(1 - x) - h, \]

for all value \( h > 0 \).

Exercise 2. Consider a first-order linear equation of the form \( x' = ax + f(t) \), where \( a \in \mathbb{R} \). Let \( y(t) \) be any solution of this equation. Prove that the general solution is \( y(t) + ce^{at} \) for \( c \in \mathbb{R} \) arbitrary.

Exercise 3. Consider a first order, linear, non autonomous equation of the form \( x' = a(t)x \).
(a). Find a formula involving integrals for the solution of this system.
(b). Prove that your formula gives the general solution of this system.

Exercise 4. First-order ODE need not have solutions that are defined for all time.
(a). Find the general solution of the equation \( x' = x^2 \).
(b). Discuss the domain over which each solution is defined.
(c). Give an example of a differential equation for which he solution satisfying \( x(0) = 0 \) is defined only for \(-1 < t < 1\).

Exercise 5. Given an ODE \( x' = p(t)x \), where \( p(t) \) is a differentiable and periodic with period \( T \). Prove that all solutions of this equation are periodic with period \( T \) if and only if

\[ \int_0^T p(t)dt = 0. \]
Exercise: Consider the ordinary differential equation \( x'(t) = Ax(t) \), where
\[
A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.
\]
(This is a slight modification of an example that we discussed in class.)

(i) Please show that the eigenvalues of \( A \) lie on the imaginary axis. Deduce from this that any solution of the differential equation \( x'(t) = Ax(t) \) must be \( 2\pi \)-periodic, i.e. \( x(t + 2\pi) = x(t) \).

(ii) Please show that there exists a positive definite \( 2 \times 2 \) matrix \( S \) such that \( SA + A^T S = 0 \).

(iii) Define a function \( Q : \mathbb{R}^2 \to \mathbb{R} \) by \( Q(x) = x^T S x \), where \( S \) denotes the matrix you found in part (ii). Please show that any solution curve is contained in a level curve of \( Q \). In other words, if \( x(t) \) is a solution of the ordinary differential equation \( x'(t) = Ax(t) \), then the function \( t \mapsto Q(x(t)) \) is constant.

(iv) Please show that every solution curve is an ellipse, and compute its principal axes. Does your result depend on the choice of the initial vector \( x(0) \)? Please explain. (Hint: Use part (iii).)