Problem Set I — Math 53h

April 2, 2015

Exercise 1. Find the general solution of the logistic differential equation with constant harvesting,

$$x' = x(1 - x) - h,$$

for all value $h > 0$.

Exercise 2. Consider a first-order linear equation of the form $x' = ax + f(t)$, where $a \in \mathbb{R}$. Let $y(t)$ be any solution of this equation. Prove that the general solution is $y(t) + ce^{at}$ for $c \in \mathbb{R}$ arbitrary.

Exercise 3. Consider a first order, linear, non autonomous equation of the form $x' = a(t)x$.
(a). Find a formula involving integrals for the solution of this system.
(b). Prove that your formula gives the general solution of this system.

Exercise 4. First-order ODE need not have solutions that are defined for all time.
(a). Find the general solution of the equation $x' = x^2$.
(b). Discuss the domain over which each solution is defined.
(c). Give an example of a differential equation for which he solution satisfying $x(0) = 0$ is defined only for $-1 < t < 1$.

Exercise 5. Given an ODE $x' = p(t)x$, where $p(t)$ is a differentiable and periodic with period T. Prove that all solutions of this equation are periodic with period T if and only if

$$\int_0^T p(t)dt = 0.$$

Exercise: Consider the ordinary differential equation $x'(t) = Ax(t)$, where

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}.$$

(This is a slight modification of an example that we discussed in class.)

(i) Please show that the eigenvalues of A lie on the imaginary axis. Deduce from this that any solution of the differential equation $x'(t) = Ax(t)$ must be 2π-periodic, i.e. $x(t + 2\pi) = x(t)$.

(ii) Please show that there exists a positive definite 2×2 matrix S such that $SA + A^T S = 0$.

(iii) Define a function $Q : \mathbb{R}^2 \to \mathbb{R}$ by $Q(x) = x^T Sx$, where S denotes the matrix you found in part (ii). Please show that any solution curve is contained in a level curve of Q. In other words, if $x(t)$ is a solution of the ordinary differential equation $x'(t) = Ax(t)$, then the function $t \mapsto Q(x(t))$ is constant.

(iv) Please show that every solution curve is an ellipse, and compute its principal axes. Does your result depend on the choice of the initial vector $x(0)$? Please explain. (Hint: Use part (iii).)