Research Statement

My Ph.D. thesis is motivated by an algebraic formulation of the long-standing Halperin-Carlsson conjecture. Namely, let k be a field and $R = k[x_1, ..., x_n]$ be a graded polynomial ring whose indeterminates x_i are assigned grading -1. Then, if (M, d) is any free, finitely generated differential-graded R-module whose total homology is non-zero and finite-dimensional, then $\mathrm{rank}_R(M)$ is conjecturally at least 2^n.

Carlsson [ref] showed that such a module is necessarily solvable; that is, there exists an ordered basis $\mathcal{B}$ for M such that the differential d, written with respect to $\mathcal{B}$, takes the form of an upper-triangular matrix squaring to zero whose entries are homogeneous polynomials in R. A further result says that when reducing modulo any maximal ideal $\mathfrak{m} \neq (x_1, \ldots, x_n)$ of R, the homology of the differential-graded $R/\mathfrak{m}$-module $M \otimes_R R/\mathfrak{m}$ is zero. By linear algebra, this implies that the $r = \mathrm{rank}_R(M)$ is even and the induced differential has rank $r/2$.

Such a differential-graded R-module M can thus be interpreted in an algebro-geometric fashion as a map f of varieties from punctured affine space $\mathbb{A}^n_k \setminus 0$ to the space X of upper-triangular square-zero matrices over k. The homogeneous degrees of the polynomials correspond to weights defining a natural k^*-action on X, and any such map f is necessarily k^*-equivariant.

My current research concerns the case when $k = \mathbb{C}$. Let V denote the algebraic variety of upper-triangular $r \times r$ square matrices over $\mathbb{C}$ that have rank $r/2$ and square to zero. I consider V as a topological space endowed with an action of the circle group S^1, defined via conjugation by diagonal matrices weighted by the polynomial degrees above. My aim is thus to bound the possible values of n for which there exists an S^1-equivariant map f from $\mathbb{C}^n \setminus 0$ to V, where the punctured n-plane $\mathbb{C}^n \setminus 0$ is equipped with the standard S^1-action of scalar multiplication.

The $H^*(BS^1)$-module structure of S^1-equivariant cohomology gives obstructions for such S^1-equivariant maps f to exist. By the contravariant functoriality of cohomology, such a map f would induce a map

$$H^*_{S^1}(V) \overset{H^*(f)}{\rightarrow} H^*_{S^1}(\mathbb{C}^n \setminus 0)\,.$$

The invariant of S^1-equivariant cohomology that I use is the vanishing order of the first Chern class, which $H^*(f)$ necessarily preserves or decreases; thus, by knowing the S^1-equivariant cohomology of these spaces, the existence of such a map can be ruled out.

As $\mathbb{C}^n \setminus 0$ is an irreducible space, it suffices to consider maps from $\mathbb{C}^n \setminus 0$ into irreducible components of V; the primary objective of my Ph.D. thesis is thus to determine the $H^*(BS^1)$-module structure of the equivariant cohomology $H^*_{S^1}(Y)$, for irreducible components Y of V. The identification of these irreducible components follows from a combinatorial formula developed by Rothbach, who gives a stratification of V into orbits of the Borel group of upper-triangular invertible matrices acting via conjugation. However, the difficulty in exploiting this structure to calculate cohomology is the intractably large number of orbits.

Building on Rothbach's work, I instead consider the orbits, fewer in number, of the larger parabolic group P_Y of invertible matrices. Here, the number and sizes of the diagonal blocks defining the parabolic group P_Y depend on the irreducible component Y. This stratification of Y into P_Y-orbits facilitates a homotopy-theoretic description of Y as the homotopy colimit of quotients of P_Y, which are simpler to study. The cohomology of the individual quotients is then amenable to computation using the Serre spectral sequence. Then, spectral sequences dealing with homotopy colimits yield information about the cohomology of Y, the homotopy colimit of these quotients.

To illustrate the nature of these decompositions, the first non-trivial cases occur when the parabolic group P_Y has diagonal blocks sizes of $n, n+k, m+k, m$ for some $n, m, k > 0$. Setting $N = \min(n, m)$, Y stratifies into N strata:

$$Y = Y_0 \cup Y_1 \cup \cdots \cup Y_N .$$

Theorem 1. *The stratum Y_i is homotopy-equivalent to the topological quotient G/H, where*

$$G = \mathrm{U}(n) \times \mathrm{U}(n+k) \times \mathrm{U}(m+k) \times \mathrm{U}(m)$$

and

$$H = \mathrm{U}(n-i) \times \mathrm{U}(k) \times \mathrm{U}(m-i) \times \mathrm{U}(i) \times \mathrm{U}(i) .$$

Here, H embeds as a subgroup of G consisting of block-diagonal matrices of the following form:

$$H \quad \to \quad \begin{pmatrix} \mathrm{U}(i) & \\ & \mathrm{U}(n-i) \end{pmatrix} \times \begin{pmatrix} \mathrm{U}(n-i) & & \\ & \mathrm{U}(i') & \\ & & \mathrm{U}(k) \end{pmatrix} \times \begin{pmatrix} \mathrm{U}(k) & & \\ & \mathrm{U}(i) & \\ & & \mathrm{U}(m-i) \end{pmatrix} \times \begin{pmatrix} \mathrm{U}(m-i) & \\ & \mathrm{U}(i') \end{pmatrix} \leq G .$$

In future work, I would like to examine how far I can extend this technique. I want to understand a larger class of matrices, and expect that my current techniques should extend to a complete analysis of the cases when the parabolic subgroup P_Y has at most 5 diagonal blocks.

There is also work to be done in extending the generality of my arguments. My present arguments apply when the ground field of these algebraic structures is the field $\mathbb{C}$ of complex numbers, as I am working in the context of classical homotopy theory and rely heavily on the existence of an inner product — one my key tools is the singular value theorem. On the other hand, there exists an algebro-geometric version of homotopy theory called $\mathbb{A}^1$-homotopy theory, along with a corresponding cohomology theory mimicking singular cohomology theory called motivic cohomology. This theory was developed by Morel and Voevodsky to work for algebraic varieties defined over any field, not just over the real and complex numbers.

Removing the inner product dependence in my homotopical descriptions of the spaces of upper-triangular square-zero matrices would be a key first step to reformulating them in a field-independent $\mathbb{A}^1$-homotopic context. With such a description in hand, the task of computing the motivic cohomology of these spaces would remain — this would be interesting in its own right, as computational methods would need to be devised to replace those used in the classical homotopy theory situation. Once generalized in this fashion, this work would then apply to the Halperin-Carlsson conjecture over any field k.