

MATH 51 — Linear Algebra Through the World of Robots

Jonathan Lee

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The aim of this note is to give hopefully more intuitive definitions of some of the linear algebra concepts covered in class. (Disclaimer: this will be edited more later, with more examples added, as I've just started typing this at 1:00am. Let me know if there are things you'd like added.)

Our setup: we're in n -dimensional space, also known as \mathbb{R}^n , and we've placed a remote-controlled robot named Karel at the origin. The remote control has k buttons, each labelled with a vector from \mathbb{R}^n — for the sake of notation, assume the i -th button is labelled with a given vector $v_i \in \mathbb{R}^n$, where $1 \leq i \leq k$. Karel has been programmed to move, changing its position by the vector v_i , each time button i is pressed. (Note that the buttons are magically engineered, in the sense that each can also be pushed a negative or fractional number of times.)

Example 1. Suppose Karel is at the position $(3, 4)$, and a button labelled with $(9, 2)$ is pushed twice. Then, Karel's new position will be

$$(3, 4) + 2 \cdot (9, 2) = (21, 8).$$

Definition 1 (Span — technical). The **span** of the vectors v_1, v_2, \dots, v_k is the set of all possible values of

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k,$$

where the c_i are arbitrarily chosen real numbers.

Definition 2 (Span — robot). The **span** of the vectors v_1, v_2, \dots, v_k is the set of all points Karel can reach by having buttons on the remote control pushed, assuming Karel starts at the origin.

Example 2. Suppose Karel is in \mathbb{R}^3 , currently at the origin. If the remote control has two buttons, $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$, then the set of points Karel can reach is the xy -plane (which is the span of v_1 and v_2).

Definition 3 (Linear independence — technical). The vectors v_1, v_2, \dots, v_k are **linearly independent** if whenever c_1, c_2, \dots, c_k are real numbers, not all zero, then the vector

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

is also not zero.

Definition 4 (Linear independence — robot). The vectors v_1, v_2, \dots, v_k are **linearly independent** if there is no way of pushing the buttons (using at least one, but not necessarily all) such that Karel ends up where it started.

Example 3. Suppose Karel lives in \mathbb{R}^2 , and there are three remote-control buttons:

$$v_1 = (2, 0), \quad v_2 = (0, 3) \quad \text{and} \quad v_3 = (4, 12).$$

Then, these vectors are linearly *dependent*, since no matter where Karel starts, it'll return to its starting position if we push button 1 once, button 2 twice and button 3 negative two times.

The two definitions above have formulations involving a single matrix, instead of numerous vectors. Keeping our same vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$, let A be the $n \times k$ matrix whose i -th column is the vector v_i .

Definition 5 (Matrix-vector product — technical). Let $x = [c_1 \ c_2 \ \dots \ c_k]^T$ be a vector in \mathbb{R}^k . Then the matrix-vector product Ax is the column vector

$$c_1v_1 + c_2v_2 + \dots + c_kv_k.$$

Definition 6 (Matrix-vector product — robot). Let $x = [c_1 \ c_2 \ \dots \ c_k]^T$ be a vector in \mathbb{R}^k . Suppose Karel starts off at the origin. Then, the matrix-vector product Ax is where Karel ends up after pushing the first button c_1 times, the second button c_2 times, and so on.

Example 4. Let Karel live in \mathbb{R}^2 , and suppose the remote control has the same buttons as in the previous example, so that

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 12 \end{bmatrix}.$$

If we push the first button once, the second button twice and the third button thrice, then Karel ends up in position $(14, 42)$. This is the same as saying that

$$A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 42 \end{pmatrix}.$$

Column space is the matrix version of span:

Definition 7 (Column space — technical). The **column space** of a matrix is the span of its columns.

Definition 8 (Column space — robot). Let Karel's remote control buttons correspond to the column vectors of a matrix A . Then, the **column space** of A is all points Karel can reach (when starting at the origin) by pushing its buttons.

We can reword one of the previous examples as follows:

Example 5. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, the column space of A is the xy -plane. One possible basis of the column space is $\{(1, 0, 0), (0, 1, 0)\}$, but there are also weirder ones, like $\{(3, 5, 0), (4, 2, 0)\}$.

Definition 9 (Nullspace — technical). The **nullspace** of a matrix A is all vectors v such that $Av = 0$.

Definition 10 (Nullspace — robot). Let Karel have remote control buttons corresponding to the matrix A . Then, the **nullspace** of A is all possible ways of pushing Karel's buttons so that it returns to its starting position. (Notationally, the vector $(3, 5, 9, \dots)$ means push the first button 3 times, the second button 5 times, the third button 9 times, and so on.)

As before, we can reword a previous example:

Example 6. Let

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 12 \end{bmatrix}.$$

No matter where Karel starts, it'll return to its starting position if we push button 1 once, button 2 twice and button 3 negative two times. In other words, $(1, 2, -2)$ is in the nullspace of A . Other vectors in the nullspace of A are $(0, 0, 0)$, $(-3, -6, 6)$ and $(\pi, 2\pi, -2\pi)$, for example. The nullspace can be described as the line in \mathbb{R}^3 parametrized (traced out) by the path $f(t) = (t, 2t, -2t)$; one possible basis is the single vector $(1, 2, -2)$, but any point on the line (other than the origin) will do.

Definition 11 (Basis — technical). Let V be a subspace of \mathbb{R}^n . The vectors v_1, v_2, \dots, v_k are a **basis** for V if they span V and are linearly independent.

Definition 12 (Basis — robot). Let V be a subspace of \mathbb{R}^n , and suppose Karel's remote control buttons are labelled v_1, v_2, \dots, v_k . Then these vectors form a basis for V if Karel cannot leave V , and furthermore, for any point in V , there is exactly one way of pushing Karel's buttons so it ends up there. (Thus, there is a correspondence between points in V and ways of pushing the buttons.)