

Coordinate systems, eigenvectors and eigenvalues

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You can find this handout, as well as others, on my website at <http://math.stanford.edu/~jlee/math51/>.

1 Systems of Coordinates

- as an application to finding parametrizations of subspaces of \mathbb{R}^n , we can introduce systems of coordinates — if $\mathcal{B} = \{v_1, v_2, \dots, v_k\}$ is a basis for a vector space V , then any vector $v \in V$ is uniquely expressible as a linear combination of the v_i ; define the *coordinates of v with respect to \mathcal{B}* to be the scalars $c_1, \dots, c_k \in \mathbb{R}$ such that

$$v = c_1v_1 + c_2v_2 + \dots + c_kv_k$$

- given a vector v in a vector space V with basis \mathcal{B} , write $[v]_{\mathcal{B}}$ to denote the vector whose entries are the coordinates of v with respect to \mathcal{B}
- if \mathcal{B} is a basis for a subspace $V \subseteq \mathbb{R}^n$, form the *change of basis matrix* C whose columns are the elements of \mathcal{B} expressed in standard coordinates; then given the coordinates $[v]_{\mathcal{B}}$ of a vector in V with respect to \mathcal{B} , we can calculate its standard coordinates from

$$v = C[v]_{\mathcal{B}}$$

- if \mathcal{B} is a basis for \mathbb{R}^n , then given a vector $v \in \mathbb{R}^n$ expressed in standard coordinates, we can calculate its coordinates with respect to \mathcal{B} from

$$[v]_{\mathcal{B}} = C^{-1}v$$

- given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a basis \mathcal{B} with change of basis matrix C , then we have

$$[T]_{\mathcal{B}} = C^{-1}[T]C$$

where $[T]$ denotes the matrix representing T with respect to standard coordinates

- a helpful way to remember the above formulas is through the following square:

$$\begin{array}{ccc}
 [v]_{\mathcal{B}} & \xrightleftharpoons[C^{-1}]{C} & [v]_{\mathcal{S}} \\
 [T]_{\mathcal{B}} \downarrow & & \downarrow [T]_{\mathcal{S}} \\
 [Tv]_{\mathcal{B}} & \xrightleftharpoons[C^{-1}]{C} & [Tv]_{\mathcal{S}}
 \end{array}$$

- we say two $n \times n$ matrices A and B are *similar* if $A = CBC^{-1}$ for some invertible matrix C ; that is, A and B represent the same linear transformation with respect to different bases
- *Proposition:*
 - similarity is an equivalence relation (satisfying symmetry, reflexivity, transitivity)
 - similar matrices have the same determinant
 - similar matrices have similar inverses
 - similar matrices have similar powers

2 Eigenvectors

- let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ a basis for \mathbb{R}^n ; if $[T]_{\mathcal{B}}$ is a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then for each $1 \leq i \leq n$, we have that

$$T(v_i) = \lambda_i v_i$$

- if $Tv = \lambda v$ for some vector $v \neq 0$ and a scalar $\lambda \in \mathbb{R}$, then we define v to be an *eigenvector* with *eigenvalue* λ for the linear transformation T
- *Proposition:* given an $n \times n$ matrix A , then a scalar $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $\lambda I_n - A$ has non-trivial nullspace if and only if $\det(\lambda I_n - A) = 0$
- *Proposition:* given an $n \times n$ matrix A with an eigenvalue λ , the set of λ -eigenvectors is the nullspace of $\lambda I_n - A$
- define the *characteristic polynomial* of an $n \times n$ matrix A to be the polynomial

$$p(\lambda) = \det(\lambda I_n - A);$$

observe that its roots are the eigenvalues of A

- given a linear transformation T , say it is *diagonalizable* if there exists a basis \mathcal{B} such that $[T]_{\mathcal{B}}$ is diagonal; such a basis is called an *eigenbasis* and consists of eigenvectors of A
- *Proposition:* if an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable