

# Coordinate systems, eigenvectors and eigenvalues

Jonathan Lee

October 21, 2008

These handouts are crudely extracted from my set of notes posted on my website at <http://math.stanford.edu/~jlee/math51/>.

## 1 Systems of Coordinates

- as an application to finding parametrizations of subspaces of  $\mathbb{R}^n$ , we can introduce systems of coordinates — if  $\mathcal{B} = \{v_1, v_2, \dots, v_k\}$  is a basis for a vector space  $V$ , then any vector  $v \in V$  is uniquely expressible as a linear combination of the  $v_i$ ; define the *coordinates of  $v$  with respect to  $\mathcal{B}$*  to be the scalars  $c_1, \dots, c_k \in \mathbb{R}$  such that

$$v = c_1v_1 + c_2v_2 + \dots + c_kv_k$$

- given a vector  $v$  in a vector space  $V$  with basis  $\mathcal{B}$ , write  $[v]_{\mathcal{B}}$  to denote the vector whose entries are the coordinates of  $v$  with respect to  $\mathcal{B}$
- if  $\mathcal{B}$  is a basis for a subspace  $V \subseteq \mathbb{R}^n$ , form the *change of basis matrix*  $C$  whose columns are the elements of  $\mathcal{B}$  expressed in standard coordinates; then given the coordinates  $[v]_{\mathcal{B}}$  of a vector in  $V$  with respect to  $\mathcal{B}$ , we can calculate its standard coordinates from

$$v = C[v]_{\mathcal{B}}$$

- if  $\mathcal{B}$  is a basis for  $\mathbb{R}^n$ , then given a vector  $v \in \mathbb{R}^n$  expressed in standard coordinates, we can calculate its coordinates with respect to  $\mathcal{B}$  from

$$[v]_{\mathcal{B}} = C^{-1}v$$

- given a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a basis  $\mathcal{B}$  with change of basis matrix  $C$ , then we have

$$[T]_{\mathcal{B}} = C^{-1}[T]C$$

where  $[T]$  denotes the matrix representing  $T$  with respect to standard coordinates

- we say two  $n \times n$  matrices  $A$  and  $B$  are *similar* if  $A = CBC^{-1}$  for some invertible matrix  $C$ ; that is,  $A$  and  $B$  represent the same linear transformation with respect to different bases
- *Proposition:*
  - similarity is an equivalence relation (satisfying symmetry, reflexivity, transitivity)
  - similar matrices have the same determinant
  - similar matrices have similar inverses
  - similar matrices have similar powers

## 2 Eigenvectors

- let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation and  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  a basis for  $\mathbb{R}^n$ ; if  $[T]_{\mathcal{B}}$  is a diagonal matrix with diagonal entries  $\lambda_1, \dots, \lambda_n$ , then for each  $1 \leq i \leq n$ , we have that

$$T(v_i) = \lambda_i v_i$$

- if  $Tv = \lambda v$  for some vector  $v \neq 0$  and a scalar  $\lambda \in \mathbb{R}$ , then we define  $v$  to be an *eigenvector* with *eigenvalue*  $\lambda$  for the linear transformation  $T$
- *Proposition:* given an  $n \times n$  matrix  $A$ , then a scalar  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A$  if and only if  $\lambda I_n - A$  has non-trivial nullspace if and only if  $\det(\lambda I_n - A) = 0$
- *Proposition:* given an  $n \times n$  matrix  $A$  with an eigenvalue  $\lambda$ , the set of  $\lambda$ -eigenvectors is the nullspace of  $\lambda I_n - A$
- define the *characteristic polynomial* of an  $n \times n$  matrix  $A$  to be the polynomial

$$p(\lambda) = \det(\lambda I_n - A);$$

observe that its roots are the eigenvalues of  $A$

- given a linear transformation  $T$ , say it is *diagonalizable* if there exists a basis  $\mathcal{B}$  such that  $[T]_{\mathcal{B}}$  is diagonal; such a basis is called an *eigenbasis* and consists of eigenvectors of  $A$
- *Proposition:* if an  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues, then it is diagonalizable