Directional Derivatives and the Gradient

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March 2, 2010

You can find this handout, as well as others, on my website at `http://math.stanford.edu/~jlee/math51/`.

- given a function $f : \mathbb{R}^n \to \mathbb{R}$, a point $p \in \mathbb{R}^n$ and a unit vector $v \in \mathbb{R}^n$, we can define $D_v f(p)$, the *directional derivative of f at a in the direction of v,* to be $g'(0)$, where we have set $g(t) = f(p + tv)$
- define the *gradient* of a function $f : \mathbb{R}^n \to \mathbb{R}$ to be the vector of partial derivatives; that is,
$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$
- directional derivatives can be expressed in terms of the gradient (which, as an overall derivative, encapsulates all the directional ones) by means of
$$D_v f(p) = \nabla f(p) \cdot v$$
  - find the directional derivatives $D_v f(p)$ for

function	point	direction
$f(x, y) = e^y \sin(x)$	$p = (\frac{\pi}{3}, 0)$	$v = \frac{1}{\sqrt{10}}(3, -1)$
$f(x, y) = \frac{1}{x^2+y^2}$	$p = (3, -2)$	$v = (1, -1)$
$f(x, y) = e^x - x^2 y$	$p = (1, 2)$	$v = (2, 1)$
$f(x, y, z) = xyz$	$p = (-1, 0, 2)$	$v = \frac{1}{\sqrt{5}}(0, 2, -1)$
$f(x, y, z) = e^{-(x^2+y^2+z^2)}$	$p = (1, 2, 3)$	$v = (1, 1, 1)$

- the tangent plane of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $p \in \mathbb{R}^n$ is given by the parametric equation
$$h(x) = f(p) + \nabla f(p) \cdot (x - p)\,;$$
in the case $n = 2$, we can use this to find non-zero vectors perpendicular to the plane using the cross-product

- a surface in $\mathbb{R}^n$ defined by the equation $f(x, y, z) = c$ has a perpendicular vector given by ∇f; hence, if ∇f is non-zero, an equation for the tangent plane at a point p is given by

  $$\nabla f(p) \cdot (v - p) = 0$$

  and the line perpendicular to the surface passing through p is given by

  $$L = \{p + t \cdot \nabla f(p) : t \in \mathbb{R}\}$$

  - find the plane tangent to the following surfaces at the given points

surface	point
$x^3 + y^3 + z^3 = 7$	$p = (0, -1, 2)$
$ze^y \cos(x) = 1$	$p = (\pi, 0, -1)$
$2xz + yz - x^2y + 10 = 0$	$p = (1, -5, 5)$
$2xy^2 = 2z^2 - xyz$	$p = (2, -3, 3)$