

Math 51 Final Exam Review Questions

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An online version of these notes is posted on my website at <http://math.stanford.edu/~jlee/math51/>.

- For each positive integer n , define A_n to be the $n \times n$ matrix with c 's on the diagonal and 1's elsewhere — for example, we would have

$$A_3 = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} c & 1 & 1 & 1 \\ 1 & c & 1 & 1 \\ 1 & 1 & c & 1 \\ 1 & 1 & 1 & c \end{bmatrix}.$$

- What is $\det A_n$? (Hint: try small values, like $n = 2, 3, 4$ and guess the pattern; also try thinking about eigenvectors and eigenvalues.) (Prove it.)
 - For which values of c is A_n invertible? or diagonalizable? (Prove it.)
 - Find the eigenvalues of A_n .
- Pick some positive number $n \geq 1$, a scalar $c \in \mathbb{R}$ and a vector $v \in \mathbb{R}^n$. Then, what is a necessary and sufficient condition on c and v so that the set

$$\{x \in \mathbb{R}^n : x \cdot v + c = 0\}$$

is a linear subspace of \mathbb{R}^n ? (Prove it.)

- Define the function $f(x, y) = x^3 + 3xy + y^3$ and take the point $p = (-1, -1)$.
 - Write down the Jacobian and Hessian matrices of f at p — that is, $Df(p)$ and $Hf(p)$.
 - Write the linear approximation to f at the point p — that is, the tangent plane.
 - Write the quadratic approximation to f at the point p .
- Consider the planes in \mathbb{R}^3 defined by $P = (3, 4, 100)^\perp$ and $Q = (4, 30, 9)^\perp$ and denote their line of intersection by L . Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by rotation about L by the angle $\pi/13$ (either orientation is fine) and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined by reflection across the plane P .
 - What is the determinant of the linear transformation $S \circ T$ (that is, the determinant of any matrix representing it)? (Prove it.)
 - Show that S is invertible and find the eigenvalues of S^{-1} . (Prove it.)
- Let x, y, z be vectors in \mathbb{R}^n with lengths 1, 2, 3, respectively. Suppose that x is parallel to (and in the same direction as) y , and x is perpendicular to z . Determine the constant(s) c such that $x + y + z$ and $x + cy + z$ are perpendicular. (Prove it.)

- Consider a matrix A and its row-reduced echelon form:

$$A = \begin{bmatrix} 1 & ? & 5 & 9 \\ 2 & ? & 6 & 10 \\ 3 & ? & 7 & 11 \\ 4 & ? & 8 & 13 \end{bmatrix} \quad \text{and} \quad \text{rref } A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the second column of A ? (Prove it.)

- Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the projection onto the plane $(1, 0, 9)^\perp$.
 - Find an eigenbasis for T .
 - Write down a matrix (or product of matrices and their inverses) in standard coordinates representing T .

- Show that if A is an $n \times n$ matrix, then there exists scalars c_0, c_1, \dots, c_n — not all zero — such that

$$\det(c_0 I_n + c_1 A + c_2 A^2 + \dots + c_n A^n) = 0.$$

(Hint: for any vector $v \in \mathbb{R}^n$, consider the set of vectors $v, Av, \dots, A^n v$ and think of why this might be useful.)

- Find all critical points of the function $2x^3 + 6xy + 3y^2$ and describe their nature.
- For a fixed number $c \in \mathbb{R}$, consider the plane

$$P_c = \{(x, y, z) : 2x + 3y - 5z = c\}.$$

- Depending on the constant c , either find a matrix A such that P_c is the nullspace of A or explain why such a matrix does not exist.
- Depending on the constant c , either find a matrix A such that P_c is the column space of A or explain why such a matrix does not exist.
- Consider the matrices:

$$C = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & 2 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} -5 & -4 & 1 \\ 1 & 1 & 0 \\ 8 & 6 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and the corresponding eigenvectors of the matrix $C^{-1}DC$. (The less you compute, the easier your life will be.)

- Find the point on the plane $x + 2y + 3z = 13$ that is closest to the point $(1, 1, 1)$.
- Consider the matrices:

$$C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 1 & a & -1 \\ 0 & b & 1 \end{bmatrix}.$$

What are a and b ?