Math 51 Final Exam Review Solutions

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An online version of these notes is posted on my website at `http://math.stanford.edu/~jlee/math51/`.

- For each positive integer n, define A_n to be the $n \times n$ matrix with c's on the diagonal and 1's elsewhere — for example, we would have

$$A_3 = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} c & 1 & 1 & 1 \\ 1 & c & 1 & 1 \\ 1 & 1 & c & 1 \\ 1 & 1 & 1 & c \end{bmatrix}.$$

  - What is $\det A_n$? (Hint: try small values, like $n = 2, 3, 4$ and guess the pattern; also try thinking about eigenvectors and eigenvalues.) (Prove it.)
  - For which values of c is A_n invertible? or diagonalizable? (Prove it.)
  - Find the eigenvalues of A_n.

Solution: Define the $n \times n$ matrix B_n to be $-A_n + cI_n$ — observe then that B is a matrix with 0's on the diagonal and -1's elsewhere, as follows:

$$B_n = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix}.$$

Recall some basic facts about eigenspaces — we say that E_α is an eigenspace of a matrix M of dimension d if the dimension of the nullspace of $\alpha I - M$ has dimension d. Moreover, by the theory covered in class, we must have that the polynomial $(\lambda - \alpha)^d$ divides the characteristic polynomial of M, which is defined to be $\det(\lambda I - M)$.

We first calculate the eigenspaces of B_n. Observe that if $\alpha = 1$, then we get the matrix

$$\alpha I_n - B_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

which row-reduces to a matrix with $n - 1$ empty rows. Thus, $\alpha I_n - B_n$ has a nullspace of dimension $n - 1$, which means that B_n has an eigenspace E_1 of dimension $n - 1$.

Similarly, observe that if $\alpha = 1-n$, then we get the matrix

$$\alpha I_n - B_n = \begin{bmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{bmatrix},$$

which has the property that all the entries in each row add up to 0. As a result, by recalling how matrix multiplication works, this means that the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$ is in the nullspace of $\alpha I_n - B_n$, so that $\alpha I_n - B_n$ has a nullspace of dimension at least 1, which means that B_n has an eigenspace E_{1-n} of dimension at least 1.

Now, by the basic facts of eigenspaces mentioned above, since B_n has an eigenspace E_1 of dimension $n-1$ and an eigenspace E_{1-n} of dimension at least 1, this means that the polynomial $(\lambda-1)^{n-1}(\lambda-(1-n))$ must divide the characteristic polynomial $p(\lambda) = \det(\lambda I_n - B_n)$ of B_n. Since B_n is an $n \times n$ matrix, the degree of $p(\lambda)$ must be n; as the degree of $(\lambda-1)^{n-1}(\lambda-(1-n))$ is also n, we must have $p(\lambda) = (\lambda-1)^{n-1}(\lambda-(1-n))$.

Since we have defined $B_n = -A_n + cI_n$, this means that $A_n = cI_n - B_n$. Doing a substitution $\lambda = c$, this shows that $\det A_n = (c-1)^{n-1}(c-(1-n))$.

We now determine the values of c for which A_n is invertible. By the general theory of determinants, we see that A_n is invertible if and only if $\det(A_n) \neq 0$. Since $\det A_n = (c-1)^{n-1}(c-(1-n))$, we see that A_n is invertible if and only if $c \neq 1, n-1$.

Moreover, by the calculations above, we see that A_n has two eigenspaces — one (E_1) of dimension $n-1$, and the other (E_{1-n}) of dimension 1. Hence, we can find an eigenbasis of A_n; that is, a basis of $\mathbb{R}^n$ whose elements are all eigenvectors of A_n. This is what it means for A_n to be diagonalizable, so we've shown that A_n is diagonalizable.

Finally, we wish to compute the eigenvalues of A_n. To do this, we must compute the characteristic polynomial of A_n, which is

$$\det(\lambda I - A_n) = \det(\lambda I_n - cI_n + B_n) = \det((\lambda - c)I_n + B_n) = (-1)^n \det((c-\lambda)I_n - B_n).$$

However, we've already computed the characteristic polynomial of B_n, which is

$$\det(\lambda I_n - B_n) = (\lambda-1)^{n-1}(\lambda-(1-n)).$$

Replacing λ by $c - \lambda$ and substituting this into the previous equation, we get

$$\det(\lambda I - A_n) = (-1)^n(c-\lambda-1)^{n-1}(c-\lambda-(1-n)) = (\lambda-(c-1))^{n-1}(\lambda-(c+1-n)).$$

Hence, the eigenvalues of A_n are $c-1$ and $c+1-n$.

- Pick some positive number $n \geq 1$, a scalar $c \in \mathbb{R}$ and a vector $v \in \mathbb{R}^n$. Then, what is a necessary and sufficient condition on c and v so that the set

  $$P_c = \{x \in \mathbb{R}^n : x \cdot v + c = 0\}$$

  is a linear subspace of $\mathbb{R}^n$? (Prove it.)

  Solution: We claim that a necessary and sufficient condition for P_c to be a linear subspace is that $c = 0$.

  Indeed, if $c = 0$, then

  $$P_c = \{x \in \mathbb{R}^n : x \cdot v = 0\} = \{x \in \mathbb{R}^n : v \cdot x = 0\} = \text{the nullspace of } v^T;$$

since nullspaces are subspaces, this shows that P_0 is a subspace.

On the other hand, if $c \neq 0$, then the zero vector 0 is not in P_c, since $0 \cdot v + c = c \neq 0$. Since all subspaces must contain the zero vector, this means P_c is not a subspace.

- Define the function $f(x, y) = x^3 + 3xy + y^3$ and take the point $p = (-1, -1)$.
  - Write down the Jacobian and Hessian matrices of f at p — that is, $Df(p)$ and $Hf(p)$.
  - Write the linear approximation to f at the point p — that is, the tangent plane.
  - Write the quadratic approximation to f at the point p.

  Solution: By taking partial derivatives, we compute the Jacobian and Hessian matrices to be

$$Df(x,y) = \begin{bmatrix} 3x^2 + 3y & 3y^2 + 3x \end{bmatrix} \quad \text{and} \quad Hf(x,y) = \begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix}.$$

  At the point $p = (-1, -1)$, these evaluate to

$$Df(p) = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \text{and} \quad Hf(p) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}.$$

  Hence, the first-degree Taylor polynomial of f centered around p is

$$\begin{aligned} p_1(x) &= f(p) + Df(p)(x-p) \\ &= 1 + \begin{bmatrix} 0 & 0 \end{bmatrix}(x-p) \\ &= 1 \end{aligned}$$

  and the second-degree Taylor polynomial of f centered around p is

$$\begin{aligned} p_2(x) &= f(p) + Df(p)(x-p) + \frac{1}{2}\left[(x-p)^T \cdot Hf(p) \cdot (x-p)\right] \\ &= 1 + \begin{bmatrix} 0 & 0 \end{bmatrix}(x-p) + \frac{1}{2}\left[(x-p)^T \cdot \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} \cdot (x-p)\right] \\ &= 1 + \frac{1}{2}\left[(x-p)^T \cdot \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} \cdot (x-p)\right]. \end{aligned}$$

- Consider the planes in $\mathbb{R}^3$ defined by $P = (3, 4, 100)^\perp$ and $Q = (4, 30, 9)^\perp$ and denote their line of intersection by L. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by rotation about L by the angle $\pi/13$ (either orientation is fine) and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by reflection across the plane P.
  - What is the determinant of the linear transformation $S \circ T$ (that is, the determinant of any matrix representing it)? (Prove it.)
  - Show that S is invertible and find the eigenvalues of S^{-1}. (Prove it.)

  Solution: Since determinant is multiplicative, we know that $\det(S \circ T) = \det(S) \cdot \det(T)$, so the problem reduces to computing $\det(S)$ and $\det(T)$.

  The linear transformation T is rotation around some line by some angle — it turns out that the actual line and angle don't matter. Since similar matrices have the same determinant, we can pick any basis we like, find a matrix representing T in that basis, and the determinant of that matrix will be the determinant of T.

Picking an orthonormal basis $\mathcal{B}$ such that the line L is spanned by the vector with $\mathcal{B}$-coordinates $[v]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, we see that T is represented by the matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and so has determinant 1.

Similarly, picking an orthonormal basis $\mathcal{B}'$ such that the plane P is spanned by the vectors with $\mathcal{B}'$-coordinates $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, we see that S is represented by the matrix

$$[S]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and so has determinant -1.

Thus, we have that

$$\det(S \circ T) = \det(S) \cdot \det(T) = \det([S]_{\mathcal{B}'}) \cdot \det([T]_{\mathcal{B}}) = -1 \cdot 1 = -1\,.$$

Finally, S is invertible because a simple calculation shows that $S^2 = S \cdot S = I$, so that S^{-1} exists and equals S. (Alternatively, $\det(S) = -1 \neq 0$ also shows that S is invertible.) Since $[S]_{\mathcal{B}'}$ is a diagonal matrix, we simply read off the diagonal entries to see that the eigenvalues of S are 1 and -1.

- Let x, y, z be vectors in $\mathbb{R}^n$ with lengths $1, 2, 3$, respectively. Suppose that x is parallel to (and in the same direction as) y, and x is perpendicular to z. Determine the constant(s) c such that $x + y + z$ and $x + cy + z$ are perpendicular. (Prove it.)

  Solution: We may pick an orthonormal basis $\mathcal{B} = \{b_1, b_2, b_3\}$ such that $b_1 = x$ and $b_2 = z/3$. Then, the coordinates of x, y, z with respect to $\mathcal{B}$ are

  $$[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [y]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad [z]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

  Since $\mathcal{B}$ is an orthonormal basis, changing coordinates preserves dot product — in particular, for any two vectors u and v, $u \cdot v = [u]_{\mathcal{B}} \cdot [v]_{\mathcal{B}}$. Applying this to the vectors $x + y + z$ and $x + cy + z$, we get that

  $$(x + y + z) \cdot (x + cy + z) = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 + 2c \\ 3 \\ 0 \end{bmatrix} = (3)(1 + 2c) + (3)(3) + (0)(0) = 3 + 6c + 9 = 6c + 12\,.$$

  Thus, the two vectors are perpendicular precisely when $c = -2$.

- Consider a matrix A and its row-reduced echelon form:

  $$A = \begin{bmatrix} 1 & ? & 5 & 9 \\ 2 & ? & 6 & 10 \\ 3 & ? & 7 & 11 \\ 4 & ? & 8 & 13 \end{bmatrix} \quad \text{and} \quad \operatorname{rref} A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

  What is the second column of A? (Prove it.)

Solution: The key fact here is that performing row operations doesn't change dependence relations between columns. In particular, a linear dependence relation holds for the columns of a matrix if and only if it holds for the row-reduced matrix.

Label the column vectors of A by v_1, v_2, v_3, v_4 and those of $\operatorname{rref} A$ by v_1', v_2', v_3', v_4'. By inspection, we see that $v_2' = v_4' - v_1' - v_3'$; by the previous remarks, this means that

$$v_2 = v_4 - v_1 - v_3 = \begin{bmatrix} 9 \\ 10 \\ 11 \\ 13 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix},$$

as desired.

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the plane $P = (1, 0, 9)^\perp$.
  - Find an eigenbasis for T.
  - Write down a matrix (or product of matrices and their inverses) in standard coordinates representing T.

Solution: Since $v_1 = (1, 0, 9)$ is orthogonal to the plane P by construction, $Tv_1 = 0$; that is, v_1 is an eigenvector of eigenvalue 0. Observe that if v is any vector in the plane P, then $Tv = v$; hence, we may choose v_2 and v_3 to be any two non-collinear vectors in the plane P. A simple choice is $v_2 = (9, 0, -1)$ and $v_3 = (0, 1, 0)$. (It turns out that these 3 vectors are in fact orthogonal, although this isn't needed.) Thus, $\mathcal{B} = \{v_1, v_2, v_3\}$ is an eigenbasis for T, and the matrix representing T with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we let C denote the change of basis matrix whose columns are given by v_1, v_2, v_3, then the change of basis formula says that the matrix in standard coordinates representing T is

$$[T] = C[T]_{\mathcal{B}}C^{-1} = \begin{bmatrix} 1 & 9 & 0 \\ 0 & 0 & 1 \\ 9 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 9 & 0 \\ 0 & 0 & 1 \\ 9 & -1 & 0 \end{bmatrix}^{-1}.$$

- Show that if A is an $n \times n$ matrix, then there exists scalars $c_0, c_1, \ldots, c_n$ — not all zero — such that

$$\det(c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n) = 0\,.$$

(Hint: for any vector $v \in \mathbb{R}^n$, consider the set of vectors $v, Av, \ldots, A^n v$ and think of why this might be useful.)

Solution: Let v be any non-zero vector, and consider the vectors $v, Av, \ldots, A^n v$. These vectors are in $\mathbb{R}^n$; since there are $n + 1$ (which is more than n) of them, they must therefore be linearly dependent by an appropriate theorem. Hence, we may take scalars $c_0, c_1, \ldots, c_n$, not all zero, such that $c_0 v + c_1 Av + \cdots + c_n A^n v = 0$. However, this shows that the matrix $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n$ has non-trivial nullspace, since

$$(c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n)v = c_0 v + c_1 Av + c_2 A^2 v + \cdots + c_n A^n v = 0$$

by linearity. Hence, $c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n$ is not invertible and therefore $\det(c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n) = 0$.

- Find all critical points of the function $f(x,y) = 2x^3 + 6xy + 3y^2$ and describe their nature.

  Solution: By taking partial derivatives, we compute the Jacobian and Hessian matrices

  $$Df(x,y) = \begin{bmatrix} 6x^2+6y & 6x+6y \end{bmatrix} \quad \text{and} \quad Hf(x,y) = \begin{bmatrix} 12x & 6 \\ 6 & 6 \end{bmatrix}.$$

  Critical points are found by solving the equation $Df(p) = 0$. The equation $6x + 6y = 0$ tells us that $x = -y$; substituting this into $6x^2 + 6y = 0$ gives us $0 = 6x^2 - 6x = 6x(x-1)$, so x is 0 or 1.

  Thus, our critical points are $(0,0)$ and $(1,-1)$. The values of the Hessian at these points are

  $$Hf(0,0) = \begin{bmatrix} 0 & 6 \\ 6 & 6 \end{bmatrix} \quad \text{and} \quad Hf(1,-1) = \begin{bmatrix} 12 & 6 \\ 6 & 6 \end{bmatrix},$$

  respectively. Since $\det Hf(0,0) < 0$ and $\operatorname{tr} Hf(0,0) > 0$, this means that $Hf(0,0)$ is an indefinite matrix and so $(0,0)$ gives a saddle point. Since $\det Hf(1,-1) > 0$ and $\operatorname{tr} Hf(1,-1) > 0$, this means that $Hf(1,-1)$ is a positive definite matrix and so $(1,-1)$ gives a local minimum.

- For a fixed number $c \in \mathbb{R}$, consider the plane

  $$P_c = \{(x,y,z) : 2x + 3y - 5z = c\}.$$

  - Depending on the constant c, either find a matrix A such that P_c is the nullspace of A or explain why such a matrix does not exist.
  - Depending on the constant c, either find a matrix A such that P_c is the column space of A or explain why such a matrix does not exist.

  Solution: Suppose that $c \neq 0$. Then, $2(0) + 3(0) - 5(0) = 0 \neq 0$, so that the zero vector is not in P_c; hence, P_c is not a linear subspace. Since the nullspace and column space of any matrix are both linear subspaces, this means that no desired matrix exists for either part.

  On the other hand, suppose that $c = 0$. Then,

  $$P_0 = \{(x,y,z) : 2x+3y-5z = 0\} = \{(x,y,z) : (2,3,-5)\cdot(x,y,z) = 0\} = \text{the nullspace of } \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}.$$

  Actually computing out this nullspace, we see that it is spanned by the vectors $(-3/2, 1, 0)$ and $(5/2, 0, 1)$, so that the nullspace is the same as the column space of the matrix

  $$\begin{bmatrix} -3/2 & 5/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Consider the matrices:

  $$C = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & 2 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} -5 & -4 & 1 \\ 1 & 1 & 0 \\ 8 & 6 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  Find the eigenvalues and the corresponding eigenvectors of the matrix $C^{-1}DC$. (The less you compute, the easier your life will be.)

  Solution: Let $B = C^{-1}DC$, and let the columns of C^{-1} be $\mathcal{B} = \{b_1, b_2, b_3\}$. By the change of basis formula, the matrix representing B with respect to the basis $\mathcal{B}$ is given by $CBC^{-1} = D$. Hence, B has two eigenvalues, 2 and 1. The 2-eigenspace is spanned by the vector $(-1,1,2)$ and the 1-eigenspace is spanned by the vectors $(-2,3,2)$ and $(1,-1,1)$.

- Find the point on the plane $x + 2y + 3z = 13$ that is closest to the point $(1, 1, 1)$.

  Solution: Observe that minimizing the function $\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$ is equivalent to minimizing the function $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$, so the problem is equivalent to minimizing $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, where we define $g(x, y, z) = x + 2y + 3z - 13$.

  Taking gradients, we obtain

  $$\nabla f(x, y, z) = 2 \begin{bmatrix} x-1 & y-1 & z-1 \end{bmatrix} \quad \text{and} \quad \nabla g(x, y, z) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

  By the method of Lagrange multipliers, critical points are given by solving the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$. Hence, from the equations

  $$2(x-1) = \lambda, \quad 2(y-1) = 2\lambda, \quad 3(z-1) = 3\lambda,$$

  we obtain

  $$x = 1 + \lambda/2, \quad y = 1 + \lambda, \quad z = 1 + 3\lambda/2.$$

  Substituting these back into the equation $13 = x + 2y + 3z$, we obtain

  $$13 = (1 + \lambda/2) + 2(1 + \lambda) + 3(1 + 3\lambda/2) = 6 + 7\lambda.$$

  Thus, $\lambda = 1$, and we have a critical point at the point $(1 + \lambda, 1 + 2\lambda, 1 + 3\lambda) = (2, 3, 4)$. Since a closest point between a line and a point always exists, this critical point must be our desired minimum.

- Consider the matrices:

  $$C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 1 & a & -1 \\ 0 & b & 1 \end{bmatrix}.$$

  What are a and b?

  Solution: By definition of inverse, we necessarily have that

  $$\begin{bmatrix} -1 & -2 & 1 \\ 1 & a & -1 \\ 0 & b & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} = C^{-1} \cdot C = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  In general, the (i, j)-th entry of a product AB of matrices is the dot product of the i-th row of A with the j-th column of B. Using this, we see that the $(2, 2)$-th entry of I is

  $$1 = (1, a, -1) \cdot (0, 1, 2) = (1)(0) + (a)(1) + (-1)(2) = a - 2,$$

  so $a = 3$. Similarly, we see that the $(3, 2)$-th entry of I is

  $$0 = (0, b, 1) \cdot (0, 1, 2) = (0)(0) + (b)(1) + (1)(2) = b + 2,$$

  so $b = -2$.