

Directional Derivatives and the Gradient

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These handouts are crudely extracted from my ongoing set of notes posted on my website at <http://math.stanford.edu/~jlee/math51/>.

- given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a point $p \in \mathbb{R}^n$ and a unit vector $v \in \mathbb{R}^n$, we can define $D_v f(p)$, the *directional derivative of f at a in the direction of v* , to be $g'(0)$, where we have set $g(t) = f(p + tv)$
- define the *gradient* of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the vector of partial derivatives; that is,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- directional derivatives can be expressed in terms of the gradient (which, as an overall derivative, encapsulates all the directional ones) by means of

$$D_v f(p) = \nabla f(p) \cdot v$$

– find the directional derivatives $D_v f(p)$ for

| function | point | direction |
|-----------------------------------|--------------------------|------------------------------------|
| $f(x, y) = e^y \sin(x)$ | $p = (\frac{\pi}{3}, 0)$ | $v = \frac{1}{\sqrt{10}}(3, -1)$ |
| $f(x, y) = \frac{1}{x^2+y^2}$ | $p = (3, -2)$ | $v = (1, -1)$ |
| $f(x, y) = e^x - x^2 y$ | $p = (1, 2)$ | $v = (2, 1)$ |
| $f(x, y, z) = xyz$ | $p = (-1, 0, 2)$ | $v = \frac{1}{\sqrt{5}}(0, 2, -1)$ |
| $f(x, y, z) = e^{-(x^2+y^2+z^2)}$ | $p = (1, 2, 3)$ | $v = (1, 1, 1)$ |

- the tangent plane of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at a point $p \in \mathbb{R}^n$ is given by the parametric equation

$$h(x) = f(p) + \nabla f(p) \cdot (x - p);$$

in the case $n = 2$, we can use this to find non-zero vectors perpendicular to the plane

- a surface in \mathbb{R}^n defined by the equation $f(x, y, z) = c$ has a perpendicular vector given by ∇f ; hence, if ∇f is non-zero, an equation for the tangent plane at a point p is given by

$$\nabla f(p) \cdot (v - p) = 0$$

– find the plane tangent to the following surfaces at the given points

| surface | point |
|----------------------------|--------------------|
| $x^3 + y^3 + z^3 = 7$ | $p = (0, -1, 2)$ |
| $ze^y \cos(x) = 1$ | $p = (\pi, 0, -1)$ |
| $2xz + yz - x^2y + 10 = 0$ | $p = (1, -5, 5)$ |
| $2xy^2 = 2z^2 - xyz$ | $p = (2, -3, 3)$ |