Taylor's Theorem

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Given a function $f(x)$ whose derivatives $f'(x), f''(x), \ldots$ all exist at $x = 0$ (or more generally, any point $x = a$), we can find its Taylor series centered around 0, which is given to be

$$\begin{aligned} f(x) &\approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0) \cdot x + f''(0)/2 \cdot x^2 + f^{(3)}(0)/6 \cdot x^3 + \cdots . \end{aligned}$$

In computing a Taylor series using summation notation, the hardest part is coming up with a closed form expression for $f^{(n)}(x)$, the n-th derivative of $f(x)$. A few steps make this easier:

1. compute a few actual derivatives $f'(x), f''(x), \ldots$
2. after having computed enough derivatives, spot a pattern and guess a general formula for $f^{(n)}(x)$
3. to be safe, prove your formula is correct via induction

What happens if we try this on the function $f(x) = \log(1 - x)$? Following the steps above, we find that

1. Differentiating madly, we find that

$$\begin{aligned} f(x) &= \log(1 - x) \\ f'(x) &= -1/(1 - x) \\ f''(x) &= -1/(1 - x)^2 \\ f^{(3)}(x) &= -2/(1 - x)^3 \\ f^{(4)}(x) &= -6/(1 - x)^4 \\ f^{(5)}(x) &= -24/(1 - x)^5 \\ f^{(6)}(x) &= -120/(1 - x)^6 \ldots \end{aligned}$$

2. It's not too hard to guess a formula for $f^{(n)}(x)$ — after differentiating enough times, you get a good feel of the pattern. Here, we hope that

$$f^{(n)}(x) = -(n-1)!/(1-x)^n \quad \text{for } n \geq 1\,.$$

3. Supposing we want to check that our formula is correct, we can prove it by induction. Indeed, assuming it holds for some value of $n \geq 1$, we have

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx} f^{(n)}(x) \\ &= \frac{d}{dx} -(n-1)!/(1-x)^n \\ &= -n!/(1-x)^{n+1} \text{ by the chain rule}\,, \end{aligned}$$

which is what our formula gives.

Since $f(x) = \log(1-x)$ and $f^{(n)}(x) = -(n-1)!/(1-x)^n$ for $n \geq 1$, we see that $f(0) = 0$ and $f^{(n)}(0) = -(n-1)!$ for $n \geq 1$. The formula for a Taylor series then shows that the Taylor series for $\log(1-x)$, centered around 0, is

$$\begin{aligned} \log(1-x) &\approx \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{-x^n}{n} \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\,. \end{aligned}$$

We can also find error estimates by Taylor's theorem, which states that the error between the n-degree Taylor polynomial $T_n(f)$, and a function f itself on the interval $[-d, d]$ is bounded by

$$\frac{M}{(n+1)!}|x|^{n+1}\,,$$

where M is a bound on $f^{(n+1)}$ on the interval $[-d, d]$.

In this example, choose $n = 3$ and the interval $[-0.3, 0.3]$. Then,

$$T_n(f) = T_3(f) = -x - \frac{x^2}{2} - \frac{x^3}{3}\,.$$

For a value of M, we bound $f^{(n+1)}(x) = f^{(4)}(x)$ on the interval $[-0.3, 0.3]$. It can be checked that the least value that works is

$$M = \frac{0.3^4}{4}\,.$$

Hence, on the interval $[-0.3, 0.3]$, our error $T_3(x) - f(x)$ is bounded by

$$\frac{M}{(3+1)!}|x|^{3+1} = \frac{0.3^4}{4 \cdot 4!}|x|^4 \text{ which is less than } \frac{0.3^4}{4 \cdot 4!} \cdot 0.3^4 \text{ on } [-0.3, 0.3]\,.$$